Topics in
Probability Theory and Stochastic Processes
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Wallis’ Formula

Rating
Mathematically Mature: may contain mathematics beyond calculus with proofs.
Section Starter Question

Can you think of a sequence or a process that approximates $\pi$? What is the intuition or reasoning behind that sequence?

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Key Concepts

1. **Wallis’ Formula** is the amazing limit

   $$
   \lim_{n \to \infty} \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2n - 1) \cdot (2n - 1) \cdot (2n + 1)} \right) = \frac{\pi}{2}.
   $$

2. One proof of Wallis’ formula uses a recursion formula developed from integration of trigonometric functions.

3. Another proof uses only basic algebra, the Pythagorean Theorem, and the formula $\pi r^2$ for the area of a circle of radius $r$.

4. Yet another proof uses Euler’s infinite product representation for the sine function.

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Vocabulary

1. **Wallis’ Formula** is the amazing limit

   $$
   \lim_{n \to \infty} \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2n - 1) \cdot (2n - 1) \cdot (2n + 1)} \right) = \frac{\pi}{2}.
   $$

2
Mathematical Ideas

Introduction

Wallis’ Formula is the amazing limit

\[
\lim_{n \to \infty} \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \ldots (2n-1) \cdot (2n-1) \cdot (2n+1)} \right) = \frac{\pi}{2}.
\]

(1)

Another way to write this is

\[
\frac{\pi}{2} = \prod_{j=1}^{\infty} \frac{(2j)^2}{(2j-1)(2j+1)}.
\]

A “closed form” expression for the product in Wallis formula is

\[
\lim_{n \to \infty} \frac{2^4n \cdot (n!)^4}{((2n)!)^2(2n+1)} = \frac{\pi}{2}
\]

or equivalently

\[
\lim_{n \to \infty} \frac{2^4n}{\left(\binom{2n}{n}\right)^2(2n+1)} = \frac{\pi}{2}.
\]

Note that Wallis Formula is equivalent to saying that the “central binomial term” has the asymptotic expression

\[
\frac{1}{2^{2n}} \binom{2n}{n} \sim \sqrt{\frac{2}{(2n+1)\pi}}.
\]

See the subsection Central Binomial below for a proof of an equivalent inequality.

In the form

\[
w_n = \prod_{j=1}^{n} \frac{(2j)^2}{(2j-1)(2j+1)} = \frac{2^{4n}}{\left(\binom{2n}{n}\right)^2(2n+1)}
\]

(2)
Figure 1: Convergence of the Wallis formula to \( \pi/2 \).
it is easy to see that the sequence \( w_n \) is increasing since \( 4n^2/(4n^2 - 1) > 1 \). This is illustrated in Figure 1.

Doing a numerical linear regression of \( \log(\pi/2 - w_n) \) versus \( \log n \) on the domain \( n = 1 \) to \( n = 30 \) indicates that \( w_n \) approaches \( \pi/2 \) at a rate which is \( O(1/n) \).

A proof using integration and recursion

**Theorem 1** (Wallis’ Formula).

\[
\lim_{n \to \infty} \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \ldots (2n - 1) \cdot (2n + 1)} \right) = \frac{\pi}{2}.
\]

(3)

**Proof.** Consider \( J_n = \int_0^{\pi/2} \cos^n(x) \, dx \). Integrating by parts with \( u = \cos^{n-1}(x) \) and \( dv = \cos(x) \) shows

\[
\int_0^{\pi/2} \cos^n(x) \, dx = (n - 1) \int_0^{\pi/2} \cos^{n-2}(x) \sin^2(x) \, dx \\
= (n - 1) \int_0^{\pi/2} \cos^{n-2}(x)(1 - \cos^2(x)) \, dx \\
= (n - 1) \int_0^{\pi/2} \cos^{n-2}(x) \, dx - (n - 1) \int_0^{\pi/2} \cos^n(x) \, dx.
\]

Gathering terms, we get \( nJ_n = (n - 1)J_{n-2} \).

Now \( J_1 = 1 \) so recursively \( J_3 = \frac{2}{3}, J_5 = \frac{2 \cdot 4}{3 \cdot 5} \) and inductively

\[
J_{2n+1} = \frac{2 \cdot 4 \cdots (2n - 2) \cdot (2n)}{1 \cdot 3 \cdots (2n - 1) \cdot (2n + 1)}.
\]

(4)

Likewise \( J_2 = \frac{\pi}{2}, \) and \( J_4 = \frac{3 \pi}{2 \cdot 4}, J_6 = \frac{3 \cdot 5 \pi}{2 \cdot 4 \cdot 6} \) and inductively

\[
J_{2n} = \frac{3 \cdot 5 \cdots (2n - 3) \cdot (2n - 1) \cdot \pi}{2 \cdot 4 \cdots (2n - 2) \cdot (2n) \cdot 2}.
\]

For \( 0 \le x \le \pi/2, 0 \le \cos(x) \le 1 \), so \( \cos^{2n}(x) \ge \cos^{2n+1}(x) \ge \cos^{2n+2}(x) \), implying in turn that \( J_{2n} \ge J_{2n+1} \ge J_{2n+2} \). Then

\[
1 \ge \frac{J_{2n+1}}{J_{2n}} \ge \frac{J_{2n+2}}{J_{2n}} = \frac{2n + 1}{2n + 2}.
\]
Hence \( \lim_{n \to \infty} \frac{J_{2n+1}}{J_{2n}} = 1 \).
That is,
\[
\lim_{n \to \infty} \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \ldots (2n - 1) \cdot (2n + 1)} \frac{2}{\pi} \right) = 1
\]
or equivalently
\[
\lim_{n \to \infty} \left( \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \ldots (2n - 1) \cdot (2n + 1)} \right) = \frac{\pi}{2}.
\]

\[
\square
\]

An elementary proof of Wallis formula

The following proof is adapted and expanded from [7]. The proof uses only basic algebra, the Pythagorean Theorem, and the formula \( \pi r^2 \) for the area of a circle of radius \( r \). Another important property used implicitly is the completeness property of the reals.

Define a sequence of numbers by
\[
s_1 = 1 \quad \text{and for } n \geq 2, \quad s_n = \frac{3}{2} \cdot \frac{5}{4} \cdots \frac{2n-1}{2n-2}.
\]
These are the reciprocals of the subsequence \( J_{2n-1} \) defined in equation (4).

The partial products of Wallis’ formula [1] with an odd number of terms in the numerator are
\[
o_n = \frac{2^2 \cdot 4^2 \cdots (2n - 2)^2 \cdot 2n}{1 \cdot 3^2 \cdots (2n - 1)^2} = \frac{2n}{s_n^2},
\]
while those with an even number of factors in the numerator are of the form
\[
e_n = \frac{2^2 \cdot 4^2 \cdots (2n - 2)^2}{1 \cdot 3^2 \cdots (2n - 3)^2 \cdot (2n - 1)} = \frac{2n - 1}{s_n^2}.
\]
Here \( e_1 = 1 \) should be interpreted as an empty product. Since \( \frac{(2n)^2}{(2n-1)(2n+1)} > 1 \), clearly \( e_n < e_{n+1} \). Since \( \frac{(2n)(2n+2)}{(2n+1)^2} < 1 \), clearly \( o_n > o_{n+1} \). Also, \( e_n < o_n \). Therefore
\[
e_1 < e_2 < e_3 < \cdots < e_n < o_n < \cdots < o_3 < o_2 < o_1
\]
for any $n$. Furthermore, for $1 \leq i \leq n$,

$$\frac{2i}{s_i^2} = a_i \geq o_n$$

and

$$\frac{2i - 1}{s_i^2} = e_i \leq e_n$$

from which it follows that

$$\frac{2i - 1}{e_n} \leq s_i^2 \leq \frac{2i}{o_n}. \quad (8)$$

For convenience, define $s_0 = 0$ so that inequality $[8]$ holds also for $i = 0$. Denote the successive difference $a_n = s_{n+1} - s_n$ so $a_0 = s_1 - s_0 = 1$ and for $n \geq 1$

$$a_n = s_{n+1} - s_n = s_n \left( \frac{2n+1}{2n} - 1 \right) = \frac{s_n}{2n} = \frac{1}{2} \cdot \frac{3}{4} \cdots \frac{2n-1}{2n}.$$  

**Lemma 2.** For the sequence $a_i$ we have the identity

$$a_i a_j = \frac{j + 1}{i + j + 1} a_i a_{j+1} + \frac{i + 1}{i + j + 1} a_{i+1} a_j \quad (9)$$

for any $i$ and $j$.

**Proof.** Make the substitutions

$$a_{i+1} = \frac{2i + 1}{2(i + 1)} a_i$$

and

$$a_{j+1} = \frac{2j + 1}{2(j + 1)} a_j$$

the right side of the proposed identity $(9)$ becomes

$$a_i a_j \left( \frac{2j + 1}{2(j + 1)} \cdot \frac{j + 1}{i + j + 1} + \frac{2i + 1}{2(i + 1)} \cdot \frac{i + 1}{i + j + 1} \right) = a_i a_j.$$
Lemma 3.

\[ 1 = a_0^2 = a_0a_1 + a_1a_0 = a_0a_2 + a_1^2 + a_2a_0 \]

\[ \cdots \]

\[ = a_0a_n + a_1a_{n-1} + \cdots + a_na_0 \]

Proof. Start from \( a_0^2 = 1 \) and repeatedly apply the identity (9). At stage \( n \) applying the identity (9) to every term the sum
\[ a_0a_n + a_1a_{n-1} + \cdots + a_na_0 \]
becomes
\[ \left(a_0a_n + \frac{1}{n}a_{1n-1}\right) + \left(\frac{n-1}{n}a_{2n-2} + \frac{2}{n}a_2a_{n-2}\right) + \cdots + \left(\frac{1}{n}a_{n-1}a_1 + a_na_0\right). \]

Collecting terms, this simplifies to \( a_0a_n + a_1a_{n-1} + \cdots + a_na_0. \]

Now divide the positive quadrant of the \( xy \)-plane into rectangles by drawing the vertical lines \( y = s_n \) and the horizontal lines \( y = s_n \) for all \( n \). Let \( R_{i,j} \) be the rectangle with lower left corner \((s_i, s_j)\) and upper right corner \((s_{i+1}, s_{j+1})\). The area of \( R_{i,j} \) is \( a_ia_j \). Thus the identity \( 1 = a_0a_n + a_1a_{n+1} + \cdots + a_na_0 \) states that the total area of the rectangles \( R_{i,j} \) for which \( i+j = n \) is 1. Let \( P_n \) be the polygonal region consisting of all rectangles \( R_{i,j} \) for which \( i+j < n \). Hence the area of \( P_n \) is \( n \).

The outer corners of \( P_n \) are the points \((s_i, s_j)\) for which \( i+j = n+1 \) and \( 1 \leq i, j \leq n \). By the Pythagorean theorem, the distance of such a point to the origin is \( \sqrt{s_i^2 + s_j^2} \). By (8) this distance is bounded above by

\[ \sqrt{\frac{2(i+j)}{o_n}} = \sqrt{\frac{2(n+1)}{o_n}}. \]

Similarly the inner corners of \( P_n \) are the points \((s_i, s_j)\) for which \( i+j = n \) and \( 0 \leq i, j \leq n \). The distance of such a point to the origin is bounded from below by

\[ \sqrt{\frac{2(i+j-1)}{e_n}} = \sqrt{\frac{2(n-1)}{e_n}}. \]
Therefore, $P_n$ contains a quarter circle of radius $\sqrt{2(n-1)/e_n}$ and is contained in a quarter circle of radius $\sqrt{2(n+1)/o_n}$. See Figure 2 for a diagram of the polygonal region $P_n$ and the corresponding inner and outer quarter circles for $n = 4$.

Since the area of a quarter circle of radius $r$ is equal to $\pi r^2$ while the area of $P_n$ is $n$, this leads to the bounds

$$\frac{(n-1)\pi}{2e_n} < n < \frac{(n+1)\pi}{2o_n} \tag{10}$$

from which it follows that

$$\frac{(n-1)\pi}{2n} < e_n < o_n < \frac{(n+1)\pi}{2n}.$$ 

Now it is clear that as $n \to \infty$, $e_n$ and $o_n$ both approach $\pi/2$.

### Wallis’ Formula and the Central Binomial Coefficient

This subsection gives a detailed proof that Wallis’ Formula gives an explicit inequality bound on the central binomial term that in turn implies the asymptotic formula for the central binomial coefficient. This proof is motivated by the derivation in [1].

Start from the expansion (2) for the Central Binomial Coefficient:

$$w_n = \prod_{j=1}^{n} \frac{(2j)^2}{(2j-1)(2j+1)} = \frac{2^{4n}}{(2n)^2(2n+1)}.$$ 

Rearrange it to

$$\left(\frac{2n}{n}\right)^2 = \frac{16^n}{2n+1} \prod_{j=1}^{n} \frac{(2j-1)(2j+1)}{(2j)^2}$$

and use the definitions (6) and (7) of the partial products of the Wallis formula to obtain

$$\left(\frac{2n}{n}\right)^2 = \frac{16^n}{(2n+1)(2n+1)\alpha_{n+1}}$$

and

$$\left(\frac{2n}{n}\right)^2 = \frac{16^n}{(2n+1)e_{n+1}}.$$
Figure 2: A diagram of the regions $R_{i,j}$ and the inner and outer quarter circles for the case $n = 4$. 
Rearrange the inequality (10) to obtain
\[
\frac{2n + 2}{\pi(n + 2)} < \frac{1}{\sigma_{n+1}} \quad \text{and} \quad \frac{1}{\epsilon_{n+1}} < \frac{2n + 2}{n\pi}.
\]

Then
\[
\frac{16^n}{(2n + 1)(2n + 1)(n + 2)\pi} < \left(\frac{2n}{n}\right)^2 < \frac{16^n}{(2n + 1)n\pi}
\]
and taking square roots and slightly rearranging again
\[
\frac{4^n}{\sqrt{(2n + 1)(\pi/2)}} \sqrt{\frac{2(n + 1)^2}{(2n + 1)(n + 2)}} < \left(\frac{2n}{n}\right) < \frac{4^n}{\sqrt{(2n + 1)(\pi/2)}} \sqrt{\frac{(n + 1)}{n}}.
\]

To simplify, take a series expansion of the square roots of the rational expressions and truncate, leaving
\[
\frac{4^n}{\sqrt{(2n + 1)(\pi/2)}} \left(1 - \frac{1}{2n}\right) < \left(\frac{2n}{n}\right) < \frac{4^n}{\sqrt{(2n + 1)(\pi/2)}} \left(1 + \frac{1}{2n}\right).
\]

A proof using the product expansion of the sine function

**Theorem 4.** The expansion of \(\sin(z)\) as an infinite product is
\[
\sin(z) = z \prod_{m=1}^{\infty} \left(1 - \frac{z^2}{m^2\pi^2}\right).
\]


Although the common proof uses complex analysis, as in the texts cited above, a proof using only elementary analysis is possible. The following proof is adapted from [4].

**Proof.** Start with the definition of the Chebyshev polynomials \(T_n(x)\) from the trigonometric identity \(\cos(nx) = T_n(\cos(x))\). Then
\[
\cos(2kx) = T_k(\cos(2x)) = T_k(1 - 2\sin^2 x).
\]
Together with the sine product identity
\[
\sin((2k + 1)x) - \sin((2k - 1)x) = 2 \sin(x) \cdot \cos(2kx)
\]
this inductively shows that there is a polynomial \(F_m\) of degree \(m\) such that
\[
\sin((2m + 1)x) = \sin(x) \cdot F_m(\sin^2(x)).
\]
(In fact, using the definition \(U_n(\cos(x)) = \frac{\sin((n+1)x)}{\sin(x)}\) for the Chebyshev polynomials of the second kind, it is easy to show that \(F_m(x) = U_{2n}(\sqrt{1 - x})\).)

Substituting \(x_k = \frac{k\pi}{2m+1}\) and noting that \(\sin((2m + 1)x_k) = 0\) shows that \(F_m\) has zeros at \(\sin^2\left(\frac{k\pi}{2m+1}\right)\) for \(k = 1, 2, \ldots, m\). These zeros are distinct, so \(F_m\) has no other zeros, then
\[
F_m(y) = F_m(0) \prod_{k=1}^{m} \left(1 - \frac{y}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right)
\]
and
\[
F_m(0) = \lim_{x \to 0} \frac{\sin((2m + 1)x)}{\sin(x)} = 2m + 1.
\]
Therefore
\[
\sin((2m + 1)x) = (2m + 1) \sin(x) \prod_{k=1}^{m} \left(1 - \frac{\sin^2(x)}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right)
\]
and changing variables
\[
\sin(x) = (2m + 1) \sin\left(\frac{x}{2m+1}\right) \prod_{k=1}^{m} \left(1 - \frac{\sin^2\left(\frac{x}{2m+1}\right)}{\sin^2\left(\frac{k\pi}{2m+1}\right)}\right).
\] \quad (11)

The goal now is to estimate the product terms. For all real \(t\), \(e^t \geq 1+t\) and therefore \(e^{-t} \leq \frac{1}{1+t}\). For \(u < 1\), the choice \(t = u/(1-u)\) gives \(e^{-u/(1-u)} \leq 1-u\). Then for every collection of numbers \(u_k \in [0, 1)\), we have
\[
1 - \sum_{k} \frac{u_k}{1-u_k} \leq e^{-\sum_k \frac{u_k}{1-u_k}} \leq \prod_{k} (1 - u_k) \leq e^{-\sum_k u_k} \leq 1. \quad (12)
\]
If in addition \(\sum_k u_k < 1\), then we also know that
\[
e^{-\sum_k u_k} \leq \frac{1}{1 + \sum_k u_k} \quad (13)
\]
and subtracting the first and third terms of (12) from 1 and using (13)

\[
\frac{\sum_k u_k}{1 + \sum_k u_k} \leq 1 - \prod_k (1 - u_k) \leq \sum_k \frac{u_k}{1 - u_k}.
\] (14)

Let \( m \) and \( N \) be positive integers with \( m > N \). Take \( x \) such that \( |x| < \frac{1}{4} N \pi \) and \( \frac{x}{\pi} \notin \mathbb{Z} \). Then define \( u_k \) by

\[
u_k = \left( \frac{\sin \left( \frac{x}{2m+1} \right)}{\sin \left( \frac{k\pi}{2m+1} \right)} \right)^2, \quad k = 1, 2, \ldots, m.
\]

Use (11) by dividing the leading factor and the first \( N \) factors onto the left side to obtain

\[
\frac{\sin(x)}{(2m+1) \sin \left( \frac{x}{2m+1} \right) \prod_{k=1}^N (1 - u_k)} = \prod_{k=N+1}^m (1 - u_k).
\]

Then use the first, third and fifth terms of (12) to see that

\[
1 - \sum_{k=N+1}^m \frac{u_k}{1 - u_k} \leq \frac{\sin(x)}{(2m+1) \sin \left( \frac{x}{2m+1} \right) \prod_{k=1}^N (1 - u_k)} \leq 1.
\] (15)

For \( 0 \leq t \leq \frac{\pi}{2} \) we have \( \sin(t) \geq \frac{2}{\pi} t \), so we see that

\[
u_k \leq \left( \frac{(2m+1) \sin \left( \frac{x}{2m+1} \right)}{2k} \right)^2 \leq \left( \frac{x}{2k} \right)^2;
\]

thus

\[
\frac{u_k}{1 - u_k} \leq \frac{x^2}{(2k)^2 - x^2} \quad \text{for} \ k > N.
\]

Hence

\[
\sum_{k=N+1}^m \frac{u_k}{1 - u_k} \leq \frac{x^2}{2N - |x|}.
\]

Thus it follows from (15) that

\[
1 - \frac{x^2}{2N - |x|} \leq \frac{\sin(x)}{(2m+1) \sin \left( \frac{x}{2m+1} \right) \prod_{k=1}^N (1 - u_k)} \leq 1.
\]
Let $m \to \infty$ so that
\[
1 - \frac{x^2}{2N - |x|} \leq \frac{\sin(x)}{x \prod_{k=1}^{N}(1 - u_k)} \leq 1.
\]
Now let $N \to \infty$ and we obtain for $\frac{x}{\pi} \notin \mathbb{Z}$
\[
\sin x = x \prod_{k=1}^{\infty} \left( 1 - \frac{x^2}{k^2 \pi^2} \right).
\]
For $\frac{x}{\pi} \in \mathbb{Z}$ this equality is also true. \hfill \Box

The following somewhat probabilistic proof of Euler’s infinite product formula is adapted from [2].

**Proof.** Start with the integral
\[
\int_{0}^{\infty} x^{t-1} \left( 1 - \frac{x}{n} \right)^n \, dx
\]
and integrate by parts once:

\[
\begin{align*}
u &= \left( 1 - \frac{x}{n} \right)^n \\
v &= \frac{1}{t} x^t \\
\frac{d}{dx}v &= x^{t-1} \\
\frac{d}{dx}u &= -\left( 1 - \frac{x}{n} \right)^{n-1} \\
\end{align*}
\]
Then
\[
\int_{0}^{\infty} x^{t-1} \left( 1 - \frac{x}{n} \right)^n \, dx = \left. \left( 1 - \frac{x}{n} \right)^n \right|_{0}^{\infty} + \int_{0}^{\infty} \frac{1}{t} x^t \left( 1 - \frac{x}{n} \right)^{n-1} \, dx
\]
\[
= \frac{1}{t} \int_{0}^{\infty} x^t \left( 1 - \frac{x}{n} \right)^{n-1} \, dx
\]
Integrate by parts again:

\[
\begin{align*}
u &= \left( 1 - \frac{x}{n} \right)^{n-1} \\
v &= \frac{1}{t+1} x^{t+1} \\
\frac{d}{dx}v &= x^t \\
\frac{d}{dx}u &= -\frac{n-1}{n} \left( 1 - \frac{x}{n} \right)^{n-2} \\
\end{align*}
\]
so
\[ \int_0^n x^{t-1} \left(1 - \frac{x}{n}\right)^n \, dx = \frac{(n-1)}{t(t+1)n} \int_0^n x^{t+1} \left(1 - \frac{x}{n}\right)^{n-2} \, dx. \]

After integration by parts \(n\) times:
\[ \int_0^n x^{t-1} \left(1 - \frac{x}{n}\right)^n \, dx = \left(\frac{n-1}{t(t+1)\ldots(t+n-1)}\right) \int_0^n x^{t+n} \, dx = \frac{n! n^t}{t(t+1)\ldots(t+n)}. \]

Then by dominated convergence it follows that
\[ \Gamma(t) = \int_0^\infty x^{t-1} e^{-x} \, dx = \lim_{n \to \infty} \frac{n! n^t}{t(t+1)\ldots(t+n)}. \]

Note that limit definition of the Gamma function is also \(\text{http://dlmf.nist.gov/5.8.E1}\). It follows that for \(-1 < t < 1\) that
\[ \Gamma(1+t)\Gamma(1-t) = \prod_{k=1}^\infty \frac{1}{1-t^2/k^2}. \]

For \(X\) and \(Y\) independent exponential random variables, both with expectation 1 we have that
\[ \mathbb{E}[X^t] = \int_0^\infty x^t e^{-x} \, dx = \Gamma(1+t) \]
and likewise \(\mathbb{E}[Y^{-t}] = \Gamma(1-t)\). Then using the independence and symmetry
\[ \Gamma(1+t)\Gamma(1-t) = \mathbb{E}[X^t] \mathbb{E}[Y^{-t}] = \mathbb{E}[(X/Y)^t] = \mathbb{E}[(X/Y)^{t}] \]

The distribution function for the random variable \(X/Y\) is
\[ \mathbb{P}[X/Y \leq u] = \mathbb{P}[Y \geq X/u] = \mathbb{E}[e^{-X/u}] = \frac{u}{u+1} \]
for \(u > 0\). Then the probability density is
\[ \frac{d}{du} \mathbb{P}[X/Y \leq u] = \frac{1}{(1+u)^2}, \quad u > 0. \]
This gives by integration by parts, for \(-1 < t < 1\), that
\[
E[(X/Y)^t] = \int_0^\infty \frac{u^t}{(u+1)^2} \, du = \int_0^\infty \frac{|t|u^{t-1}}{u+1} \, du
\]

The last integral is known to be \(\pi t / \sin(\pi t)\). For example, this is formula 613, page 445 in the 18th Edition of the CRC Standard Mathematical Tables. It can also be accomplished with contour integration in the complex plane
\[
\int_0^{+\infty} \frac{|t|u^{t-1}}{u+1} \, du \quad 2\pi i(-1)^{|t|-1} + \int_{+\infty}^{0} \frac{|t|u^{2\pi i|t|-1}}{u+1} \, du = 0
\]
so
\[
\int_0^{+\infty} \frac{|t|u^{t-1}}{u+1} \, du \quad \left(1-e^{2\pi i|t|}\right) = -2\pi i e^{\pi i|t|}.
\]
Thus
\[
\int_0^{+\infty} \frac{|t|u^{t-1}}{u+1} \, du = \frac{|t|2\pi i e^{\pi i|t|}}{e^{2\pi i|t|}-1} = \frac{\pi |t|}{\sin(\pi |t|)} = \frac{\pi t}{\sin(\pi t)}.
\]
Hence, for \(-1 < t < 1\)
\[
\Gamma(1+t)\Gamma(1-t) = E[(X/Y)^t] = \pi t / \sin(\pi t).
\]
A standard integration-by-parts gives the fundamental Gamma function recursion \(\Gamma(t) = \Gamma(1+t)/t\). Using this to define \(\Gamma(t)\) for \(t < 0\), it follows that for all real \(t\),
\[
\sin(\pi t) = \frac{\pi t}{\Gamma(t)\Gamma(1-t)}.
\]
Combining all results above,
\[
\sin(\pi t) = \pi t \prod_{k=1}^{\infty} \left(1 - \frac{t^2}{k^2}\right).
\]

\textbf{Corollary 1} (Wallis’ Formula).
\[
\lim_{n \to \infty} \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots (2n) \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot 5 \ldots (2n-1) \cdot (2n-1) \cdot (2n+1)} = \frac{\pi}{2}.
\]
\textit{Proof.} Substitute \(z = \pi/2\) in the product expansion of \(\sin(z)\). \qed

\textit{Remark.} In \cite{5} Ciaurri uses Tannery’s Theorem for Infinite Products, a trig identity for cotangent and tangent, and Wallis formula to provide an elementary proof of product expansion of the sine function. In this sense, the product expansion of the sine function is equivalent to Wallis’ formula.
Sources

This section is adapted from a sketch of the proof of Wallis’ Formula in Kazarinoff [3] and the short note by Wästlund [7]. The elementary proofs of the sine product are from [4] and [2].

Problems to Work for Understanding

1. Show that

\[
\frac{2^{4n} \cdot (n!)^4}{((2n)!)^2(2n + 1)} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \ldots (2n - 1) \cdot (2n + 1)}.
\]

2. Numerically find the rate at which the sequence

\[
\frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \ldots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \ldots (2n - 1) \cdot (2n + 1)}
\]

approaches π/2 by calculating values and using regression.

3. Use induction to prove

\[
J_{2n+1} = \frac{(2n) \cdot (2n - 2) \ldots 4 \cdot 2}{(2n + 1) \cdot (2n - 1) \ldots 3 \cdot 1}
\]

and

\[
J_{2n} = \frac{(2n - 1) \cdot (2n - 3) \ldots 3 \cdot \pi}{(2n) \cdot (2n - 2) \ldots 4 \cdot 2 \cdot 2}.
\]

4. The simple proof of the sine product formula uses the polynomial \( F_m \) of degree \( m \) such that

\[
\sin((2m + 1)x) = \sin(x) \cdot F_m(\sin^2(x)).
\]

Explicitly find \( F_0(x), F_1(x), F_2(x), F_3(x), F_4(x) \) and plot them on \([-1, 1]\).
5. Give a detailed proof that
\[
\int_0^n x^{t-1} \left(1 - \frac{x}{n}\right)^n \, dx = \frac{n! n^t}{t(t+1) \ldots (t+n)}.
\]

6. Provide a careful and detailed proof that if \(X\) and \(Y\) are independent exponential random variables, both with expectation 1 we have that
\[
\mathbb{E}\left[(X/Y)^t\right] = \mathbb{E}\left[(X/Y)^{|t|}\right].
\]

7. Provide a detailed proof using contour integration in the complex plane to show that
\[
\int_0^\infty \frac{|t|^{|t|}-1}{u+1} \, du = \frac{|t|2\pi i e^{\pi i |t|}}{e^{2\pi i |t|}-1} = \frac{\pi t}{\sin(\pi t)}.
\]

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**Reading Suggestion:**

**References**


[4] R. A. Kortram. Simple proofs for \(\sum_{k=1}^\infty \frac{1}{k^2} = \frac{\pi^2}{6}\) and \(\sin x = x \prod_{k=1}^\infty \left(1 - \frac{x^2}{k^2 \pi^2}\right)\). *Mathematics Magazine*, 69(2):123–125, April 1996.


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**Outside Readings and Links:**

1.

2.

3.

4.

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