Topics in
Probability Theory and Stochastic Processes
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Stirling’s Formula by Euler-Maclaurin Summation

Rating
Mathematicians Only: prolonged scenes of intense rigor.
Section Starter Question

How would you evaluate or even approximate \(^{2n\choose n}\) for large values of \(n\), say \(n \geq 100\) ?

Key Concepts

1. **Stirling’s Formula**, also called Stirling’s Approximation, is the asymptotic relation
   \[
   n! \sim \sqrt{2\pi n^{n+1/2}}e^{-n}.
   \]
   2. The formula is useful in estimating large factorial values, but its main mathematical value is in limits involving factorials.
   3. A first-order **Euler-Maclaurin Summation Formula** is
   \[
   f(0)+f(1)+\cdots+f(n) = \int_0^n f(t) \, dt + \frac{1}{2} (f(0) + f(n)) + \int_0^n (t-[t]-1/2)f'(t) \, dt
   \]

Vocabulary

1. **Stirling’s Formula**, also called Stirling’s Approximation, is the asymptotic relation
   \[
   n! \sim \sqrt{2\pi n^{n+1/2}}e^{-n}.
   \]
   2. A first-order **Euler-Maclaurin Summation Formula** is
   \[
   f(0)+f(1)+\cdots+f(n) = \int_0^n f(t) \, dt + \frac{1}{2} (f(0) + f(n)) + \int_0^n (t-[t]-1/2)f'(t) \, dt
   \]
Mathematical Ideas

Stirling’s Formula

Stirling’s Formula, also called Stirling’s Approximation, is the asymptotic relation

\[ n! \sim \sqrt{2\pi n^{n+1/2}} e^{-n}. \]

The formula is useful in estimating large factorial values, but its main mathematical value is in limits involving factorials. Another attractive form of Stirling’s Formula is

\[ n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n. \]

An improved inequality version of Stirling’s Formula is

\[ \sqrt{2\pi n^{n+1/2}} e^{-n+1/(12n+1)} < n! < \sqrt{2\pi n^{n+1/2}} e^{-n+1/(12n)}. \] (1)

See Stirling’s Formula.

James Stirling, a Scottish mathematician born in 1692 near Stirling Scotland, received much of his mathematical education in Oxford. He was a friend and correspondent with Nicholaus (I) Bernoulli, Newton, De Moivre, Euler, and Maclaurin. He wrote extensively on methods for accelerating summations of series, and in 1730 published the asymptotic result that bears his name in his book Methodus Differentialis on this subject. See MacTutor History of Mathematics.

According to Diaconis and Freedman, [1], De Moivre proved a version of Stirling’s Formula without the asymptotic constant \( \sqrt{2\pi} \) in 1730 while deriving what we now call the DeMoivre-Laplace Theorem for the normal approximation to the binomial distribution. Stirling also gave a proof of what is now called Stirling’s Formula using Euler-Maclaurin summation and found the constant \( \sqrt{2\pi} \) using Wallis’ Formula. This is the plan for the proof in this section.
An Easy but Crude Discovery Result

See Stirling’s Formula:

\[
\log(n!) = \log(1) + \log(2) + \cdots + \log(n)
\]

\[
= \sum_{k=1}^{n} \log(k)
\]

\[
\approx \int_{1}^{n} \log(x) \, dx
\]

\[
= [x \log(x) - x]_{1}^{n}
\]

\[
= n \log(n) - n + 1
\]

\[
\approx n \log(n) - n.
\]

Although this gives the general functional form, it does not automatically give the “asymptotic factor” \( \sqrt{2\pi n} \). The approximation comes from replacing the sum with an integral, but this can be justified precisely and rigorously with the Euler-Maclaurin sum formula.

An analytic proof using Euler-Maclaurin Summation and Wallis’ Formula

This proof is adapted and expanded from Todd, \([2]\), pages 73-75 with improvements suggested by Jamie Radcliffe and Stephen Hartke.

**Lemma 1** (Trapezoidal Approximation). If \( f \) is a function such that \( f'(x) \) is continuous on \([r-1, r]\)

\[
\frac{1}{2} (f(r - 1) + f(r)) = \int_{r-1}^{r} f(t) \, dt + \int_{r-1}^{r} (t - (r - 1) - 1/2)f'(t) \, dt.
\]

**Proof.** Starting from \( \int_{r-1}^{r} (t - (r - 1) - 1/2)f'(t) \, dt \), integrate by parts to obtain

\[
\int_{r-1}^{r} \left( t - (r - 1) - \frac{1}{2} \right) f'(t) \, dt = \frac{1}{2} f(r) + \frac{1}{2} f(r - 1) - \int_{r-1}^{r} f(t) \, dt.
\]

Then transpose \( \int_{r-1}^{r} f(t) \, dt \) to the other side. \qed
Remark. The left side in the lemma is the one-interval Trapezoidal Approximation of the area under $f(t)$ over $[r-1, r]$. Then the term $\int_{r-1}^{r} (t - (r - 1) - \frac{1}{2}) f'(t) \, dt$ is an expression for the error in approximation.

Remark. The term $(t - (r - 1) - 1/2)$ appears unmotivated, although it appears naturally in the alternative proof of Lemma 2. On the other hand, integration-by-parts to obtain the Trapezoidal Approximation requires a linear function which is $1/2$ at $r$ and $-1/2$ at $r - 1$. Those requirements mean $t - (r - 1) - 1/2$ is the natural choice.

Remark. Assume that $f(t)$ is sufficiently differentiable on $[r - 1, r]$ so that we can expand $f'(t)$ in a Taylor polynomial $f'(t) = f'((r - 1) + 1/2) + (t - (r - 1) - 1/2)f''(c_t)$ where $c_t$ is between $t$ and $(r - 1) + 1/2$. Then

$$\int_{r-1}^{r} (t - (r - 1) - 1/2)f'(t) \, dt$$

$$= \int_{r-1}^{r} (t - (r - 1) - 1/2)(f'(1/2) + (t - (r - 1) - 1/2)f''(c_t)) \, dt$$

$$= \int_{r-1}^{r} (t - (r - 1) - 1/2)^2 f''(c_t) \, dt$$

Then

$$\left| \int_{r-1}^{r} (t - (r - 1) - 1/2)f'(t) \, dt \right|$$

$$\leq \int_{r-1}^{r} (t - (r - 1) - 1/2)^2 \, dt \max_{t \in [r-1, r]} |f''(t)|$$

$$= \frac{1}{12} \max_{t \in [r-1, r]} |f''(t)|.$$ 

This is the standard error estimate for the Trapezoidal Approximation derived in numerical analysis texts. The Trapezoidal Approximation and the error estimate are usually derived by linear interpolation through the endpoints of integration, also known as Lagrange Interpolation, or the Newton-Cotes formula.

Lemma 2. Suppose

1. $f$ is a function such that $f'(x)$ is continuous on $[0, n]$ for some integer $n$. 

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2. Let \( B_1(t) \) be the periodic extension with period 1 of the function which is \( t - 1/2 \) for \( t \in [0, 1] \).

Then

\[
f(0) + f(1) + \cdots + f(n) = \int_0^n f(t) \, dt + \frac{1}{2} (f(0) + f(n)) + \int_0^n B_1(t) f'(t) \, dt
\]

**Remark.** This is a simple version of the **Euler-Maclaurin Summation Formula.**

**Proof.** Using Lemma 1

\[
\frac{1}{2} (f(0) + f(1)) = \int_0^1 f(t) \, dt + \int_0^1 (t - 1/2) f'(t) \, dt.
\]

Then

\[
f(0) + \frac{1}{2} f(1) = \int_0^1 f(t) \, dt + \frac{1}{2} f(0) + \int_0^1 (t - 1/2) f'(t) \, dt
\]

and the result follows by inductively applying the result of Lemma 1 on intervals \([1, 2], \ldots, [n-1, n]\) and finally adding \( 1/2 f(n) \) to both sides. Note that \( B_1(t) = t - (r-1) - 1/2 \) on \([r-1, r]\).

**Alternative Direct Proof.** Since \( f'(x) \) is continuous on \([0, n]\)

\[
\int_{r-1}^r f'(t) \, dt = f(r) - f(r-1), \quad r = 1, 2, \ldots, n.
\]

If we multiply this by \( r \) and sum we get

\[
\sum_{r=1}^n \int_{r-1}^r r f'(t) \, dt = \sum_{r=1}^n r(f(r) - f(r-1))
\]

\[
= \sum_{r=1}^n (r f(r) - (r-1) f(r-1)) - \sum_{r=1}^n f(r-1)
\]

\[
= n \cdot f(n) - 0 \cdot f(0) - (f(0) + f(1) + \cdots + f(n-1))
\]

\[
= -(f(0) + f(1) + \cdots + f(n)) + (n+1) f(n)
\]
Note that we can write \( r = \lfloor t \rfloor + 1 \) in the integral, so that after rearranging we have
\[
f(0) + f(1) + \cdots + f(n) = (n + 1)f(n) - \int_0^n (\lfloor t \rfloor + 1)f'(t) \, dt \quad (2)
\]
Next, by integration by parts we have
\[
\int_0^n t f'(t) \, dt = n \cdot f(n) - \int_0^n f(t) \, dt . \quad (3)
\]
Then solving for \( nf(n) \) in (2) and substituting it into (3), obtain
\[
f(0) + f(1) + \cdots + f(n) = \int_0^n f(t) \, dt + f(n) - \int_0^n (n - \lfloor t \rfloor + 1)f'(t) \, dt.
\]
Take the last term and rearrange it as
\[
- \int_0^n (-t + \lfloor t \rfloor + 1)f'(t) \, dt = \int_0^n (t - \lfloor t \rfloor - 1/2 - 1/2)f'(t) \, dt
\]
and finally use the Fundamental Theorem of Calculus on the last term. Thus
\[
f(0) + f(1) + \cdots + f(n) = \int_0^n f(t) \, dt + \frac{1}{2} (f(0) + f(n))/2 + \int_0^n (t - \lfloor t \rfloor - 1/2)f'(t) \, dt.
\]
or equivalently using the definition of \( B_1(t) \)
\[
f(0) + f(1) + \cdots + f(n) = \int_0^n f(t) \, dt + \frac{1}{2} (f(0) + f(n)) + \int_0^n B_1(t)f'(t) \, dt
\]

\textbf{Lemma 3.}

\[
\log(n!) = \left( n + \frac{1}{2} \right) \log(n) - n + 1 + \int_1^\infty \frac{B_1(x)}{x} \, dx - \epsilon_n
\]

where
\[
\epsilon_n = \int_n^\infty \frac{B_1(x)}{x} \, dx
\]
and \( \epsilon_n \rightarrow 0 \) as \( n \rightarrow \infty \).
Proof. Using Lemma 2 with \( f(x) = \log(1 + x) \) on \([0, n - 1]\)

\[
\log(1) + \log(2) + \cdots + \log(n) = \int_0^{n-1} \log(1 + x) \, dx + \frac{1}{2} (\log(1) + \log(n)) + \int_0^{n-1} \frac{B_1(x)}{1 + x} \, dx.
\]

After rearranging and the change of variables \( y = 1 + x \) on the last integral
this becomes

\[
\log(n!) = n \log(n) - n + 1 + \frac{1}{2} \log(n) + \int_1^n \frac{B_1(x)}{x} \, dx.
\]

Looking back at the definition \( B_1(x) = x - \lfloor x \rfloor - \frac{1}{2} \), we can regard \( \int_1^n \frac{B_1(x)}{x} \, dx \) as the sum of the first \( 2(n - 1) \) terms of an alternating series with terms of decreasing magnitude. Hence \( \int_1^\infty \frac{B_1(x)}{x} \, dx \) converges. Then we can write

\[
\log(n!) = n \log(n) - n + 1 + \frac{1}{2} \log(n) + \int_1^\infty \frac{B_1(x)}{x} \, dx - \epsilon_n
\]

where

\[
\epsilon_n = \int_n^\infty \frac{B_1(x)}{x} \, dx
\]

and \( \epsilon_n \to 0 \) as \( n \to \infty \). \( \square \)

Lemma 4 (Wallis’ Formula).

\[
\lim_{n \to \infty} \frac{2^{4n} \cdot (n!)^4}{((2n)!)^2(2n + 1)} = \frac{\pi}{2}
\]

Proof. See the proofs in [Wallis Formula]. \( \square \)

Theorem 5 (Stirling’s Formula).

\[
\log(n!) = \left( n + \frac{1}{2} \right) \log(n) - n + \log(\sqrt{2\pi}) - \int_n^\infty \frac{B_1(x)}{x} \, dx
\]

or equivalently

\[
n! \sim \left( \frac{n}{e} \right)^n \sqrt{2\pi n}
\]

as \( n \to \infty \).
Proof. For convenience, let
\[ J = \int_1^\infty \frac{B_1(x)}{x} \, dx. \]

Start with Wallis’ Formula and take logarithms, also use Lemma 3 on the factorials:
\[
4n \log(2) + 4 \left( n + \frac{1}{2} \right) \log(n) - 4n + 4 + 4J - 4\epsilon_n \\
- 2 \left[ \left( 2n + \frac{1}{2} \right) \log(2n) - 2n + 1 + J - \epsilon_{2n} \right] - \log(2n + 1) \to \log \left( \frac{\pi}{2} \right) \tag{4}
\]

Expand, gather and cancel like terms and obtain
\[
\log(n) + 2 + 2J - 4\epsilon_n - \log(2) - 2\epsilon_{2n} - \log(2n + 1) \to \log \left( \frac{\pi}{2} \right)
\]

Note that \(-\log(2n + 1) - \log(n) = -\log(2 + 1/n) \to -\log(2)\) as \(n \to \infty\) so
\[
2 + 2J = \log \left( 2\pi \right)
\]

and rearranging \(J = \log \left( \sqrt{2\pi} \right) - 1\).

Therefore, from Lemma 3 we can write
\[
\log(n!) = \left( n + \frac{1}{2} \right) \log(n) - n + \log(\sqrt{2\pi}) - \int_n^\infty \frac{B_1(x)}{x} \, dx
\]

which after exponentiation can be written in the more familiar form:
\[
n! \sim \sqrt{2\pi n} \left( \frac{n}{e} \right)^n.
\]

\( \square \)

**Corollary 1** (Error Estimate on Stirling’s Formula).
\[
0 < \log(n!) - \left[ \left( n + \frac{1}{2} \right) \log(n) - n + \log(\sqrt{2\pi}) \right] < \frac{1}{12 n - 1/2}
\]

or equivalently
\[
\sqrt{2\pi} n^{n+1/2} e^{-n} < n! < \sqrt{2\pi} n^{n+1/2} e^{-n+1/(12(n-1/2))}.
\]
Remark. This estimate is not as strong as
\[ \sqrt{2\pi n} n^{n+1/2} e^{-n+1/(12n+1)} < n! < \sqrt{2\pi n} n^{n+1/2} e^{-n+1/(12n)}. \]
quoted above in equation [1].

Proof. The proof makes the estimate on
\[ \epsilon_n = \int_n^\infty \frac{B_1(x)}{x} \, dx. \]
Start with
\[ \epsilon_n = \sum_{r=n}^\infty (\epsilon_r - \epsilon_{r+1}) \]
and then estimate
\[ \epsilon_r - \epsilon_{r+1} = \int_r^{r+1/2} \frac{B_1(x)}{x} \, dx + \int_{r+1/2}^{r+1} \frac{B_1(x)}{x} \, dx \]
\[ = - \int_{1/2}^0 \frac{B_1(r+1/2-y)}{r+1/2-y} \, dy + \int_0^{1/2} \frac{B_1(r+1/2+z)}{r+1/2+z} \, dz \]
\[ = \int_0^{1/2} \frac{B_1(1/2-y)}{r+1/2-y} + \frac{B_1(1/2+y)}{r+1/2+y} \, dy \]
\[ = \int_0^{1/2} \frac{-y}{r+1/2-y} + \frac{y}{r+1/2+y} \, dy \]
\[ = -2 \int_0^{1/2} \frac{y^2}{(r+1/2)^2-y^2} \, dy \]
Therefore
\[ -\epsilon_n = \sum_{r=n}^\infty \int_0^{1/2} \frac{2y^2}{(r+1/2)^2-y^2} \, dy \]
\[ < \sum_{r=n}^\infty r^{-2} \int_0^{1/2} 2y^2 \, dy \]
\[ = \frac{1}{12} \sum_{r=n}^\infty r^{-2} \]
Now
\[ r^{-2} < \left( r^2 - \frac{1}{4} \right)^{-1} = \left( r - \frac{1}{2} \right)^{-1} - \left( r + \frac{1}{2} \right)^{-1} \]
so
\[
0 < -\epsilon_n < \frac{1}{12} \sum_{r=n}^{\infty} \left( r - \frac{1}{2} \right)^{-1} - \left( r - \frac{1}{2} \right)^{-1} < \frac{1}{12} \left( n - \frac{1}{2} \right)^{-1}
\]

\[\Box\]

Sources

Problems to Work for Understanding

1. Make a table of values of \( n! \), \( \sqrt{2\pi n^{n+1/2}e^{-n}} \), \( \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n+1)}} \) and \( \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n)}} \) for \( n \) from 1 to 15. Compute the relative error between \( n! \) and \( \sqrt{2\pi n^{n+1/2}e^{-n}} \) for each value of \( n \). Likewise compute the relative error between \( n! \) and \( \sqrt{2\pi n^{n+1/2}e^{-n+1/(12n)}} \) for each value of \( n \).

2. Using a computer algebra system compute 100! and the Stirling’s Formula approximation. Calculate the absolute error and the relative error between the two values.

3. Show that
\[
\sum_{r=1}^{n} r(f(r) - f(r - 1)) = -(f(0) + f(1) + \cdots + f(n)) + (n + 1)f(n)
\]
by using “summation by parts”.

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4. Establish and prove the Trapezoidal Approximation and its error for approximating $\int_a^b f(t) \, dt$ given the function values at $a = x_0, x_1, x_2, \ldots, x_n = b$ where $x_i - xi - 1 = h = (b - a) / n$ uniformly for all $i$.

5. Show that

$$\frac{2^{4n} \cdot (n!)^4}{((2n)!)^2(2n+1)} = \frac{2 \cdot 2 \cdot 4 \cdot 4 \cdot 6 \cdot 6 \cdots (2n) \cdot (2n)}{1 \cdot 3 \cdot 3 \cdot 5 \cdot 5 \cdots (2n-1) \cdot (2+1)}.$$ 

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**Reading Suggestion:**

**References**


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**Outside Readings and Links:**

1.

2.

3.

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