Topics in
Probability Theory and Stochastic Processes
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The Sum of Independent Normal Random Variables is Normal

Rating
Student: contains scenes of mild algebra or calculus that may require guidance.
Section Starter Question

What must be known to find the distribution of the sum if two normal random variables are not independent?

Key Concepts

1. We can prove that the sum of independent normal random variables is again normally distributed by using the rotational invariance of the bivariate normal distribution.

2. The Central Limit Theorem provides a heuristic explanation of why the sum of independent normal random variables is normally distributed.

Vocabulary

1. The bivariate normal distribution is

\[ f(z_1, z_2) = \frac{\exp\left(-\frac{(z_1^2 + z_2^2)}{2}\right)}{2\pi}. \]

the joint density function of two independent standard random variables.
Mathematical Ideas

The proof that the sum of independent random variables is normal usually occurs in one of two forms. One is the direct proof using the fact that the distribution of the sum of independent random variables is the convolution of the distributions of the two independent random variables. The computation is tedious. The computation is also not illuminating about why the sum of independent normal variables is normal.

The second proof uses the fact that the moment generating function (mgf) of the sum of independent random variables is the product of the respective moment generating functions. After computing the mgf of a normal and taking the product of two mgfs, we see that the product is again the mgf of a normal random variable. Then the proof follows by using the uniqueness theorem for an mgf, that is, the fact that the moment generating function is uniquely determined by the distribution.

The rotation proof that a sum of independent normals is normal

This section is a summary, explanation, and review of the article by Eisenberg and Sullivan [2].

Lemma 1. $X$ is normal with mean $\mu$ and variance $\sigma^2$ if and only if it can be written as $X = \mu + \sigma^2 Z$ where $Z$ is standard normal with mean 0 and variance 1.

Proof. Left as an exercise.

Lemma 2. The joint density of two independent standard normal random variables is rotation invariant.

Proof. Take two independent standard normal random variables $Z_1$ and $Z_2$. Taking the product of the distributions, the joint density function of the two random variables is

$$f(z_1, z_2) = \frac{\exp\left(-\frac{z_1^2 + z_2^2}{2}ight)}{2\pi}.$$ 

This distribution is rotationally invariant. This means that the function has the same value for all points equally distant from the origin. This is obvious algebraically from the form of the variables $z_1^2 + z_2^2$, or from the
Let $X_1, X_2$ be two independent normal random variables. Since for any random variables the mean of the sum is the sum of the means, we may as well take the independent random variables to have mean 0. So take $X_1$ to be normal with mean 0 and variance $\sigma_1^2$ and $X_2$ to be normal with 0 and variance $\sigma_2^2$. Now consider the distribution of the sum $X_1 + X_2$ which has the same distribution as $\sigma_1 Z_1 + \sigma_2 Z_2$. Apply Lemmas 2 to an arbitrary half-plane. Hence

$$\mathbb{P} \left[ X_1 + X_2 \leq t \right] = \mathbb{P} \left[ \sigma_1 Z_1 + \sigma_2 Z_2 \leq t \right] = \mathbb{P} \left[ (Z_1, Z_2) \in A \right],$$

where the half-plane is $A = \{(z_1, z_2) | \sigma_1 z_1 + \sigma_2 z_2 \leq t \}$. The boundary line $\sigma_1 z_1 + \sigma_2 z_2 = t$ of the half-plane $A$ lies at a distance $d = |t|/\sqrt{\sigma_1^2 + \sigma_2^2}$ from the origin. It follows that the half-plane $A$ can be rotated into the set

$$T(A) = \left\{ (z_1, z_2) | z_1 \leq \frac{t}{\sqrt{\sigma_1^2 + \sigma_2^2}} \right\}.$$

See Figure 2 for the case when $t > 0$, so the half-plane contains the origin. See Figure 3 for the case when $t < 0$, so the origin is not in the half-plane.

Now it is easy to calculate

$$\mathbb{P} \left[ (Z_1, Z_2) \in T(A) \right] = \int \int_{T(A)} \frac{\exp\left(\frac{-(z_1^2 + z_2^2)}{2}\right)}{2\pi} dz_1 \, dz_2.$$
Thus $P[X_1 + X_2 \leq t] = P[Z_1 \leq \frac{t}{\sqrt{\sigma_1^2 + \sigma_2^2}}] = P[\sqrt{\sigma_1^2 + \sigma_2^2} Z_1 \leq t]$. Therefore $X_1 + X_2$ is normal with mean 0 and variance $\sqrt{\sigma_1^2 + \sigma_2^2}$.

This proof is elementary, self-contained, conceptual, uses geometric ideas and requires almost no computation.

**Alternative rotation proof that a sum of independent normals is normal**

This section is an explanation and review of the article by Eisenberg [1].

Using Lemma [1] given $X_1 = a_1 + b_1 Z_1$ and $X_2 = a_2 + b_2 Z_2$ are independent normal random variables, then we can assume that $Z_1$ and $Z_2$ are independent standard normal random variables. Then $X_1 + X_2$ will be normal if and only if $Y_1 = b_1 Z_1 + b_2 Z_2$ is normal. Since multiples of normal random variables are normal (again by Lemma [1]), assume without loss of generality that $b_1^2 + b_2^2 = 1$. Also, let $Y_2 = -b_2 Z_1 + b_1 Z_2$. Then

$$
\begin{pmatrix}
Y_1 \\
Y_2 
\end{pmatrix} =
\begin{pmatrix}
b_1 & b_2 \\ -b_2 & b_1 
\end{pmatrix}
\begin{pmatrix}
Z_1 \\
Z_2 
\end{pmatrix} = U
\begin{pmatrix}
Z_1 \\
Z_2 
\end{pmatrix}
$$

where $U$ is a rotation matrix. It follows from Lemma [2] that $Y_1$ and $Y_2$ have the same joint distribution as $Z_1$ and $Z_2$. That is, they are independent standard normal random variables.
A heuristic explanation

It is possible to explain heuristically why the sum of independent normal random variables is normal, using the Central Limit Theorem as given. Recall that the Central Limit Theorem says that if $X_1, X_2, \ldots$ is a sequence of independent, identically distributed random variables with mean 0 and variance 1, then

$$P \left[ \frac{X_1 + \cdots + X_n}{\sqrt{n}} \leq t \right] \xrightarrow{D} P[Z \leq t]$$

where $Z$ is normally distributed with mean 0 and variance 1. Then

$$P \left[ \frac{X_1 + \cdots + X_n}{\sqrt{n}} \leq t \right] \xrightarrow{D} P[Z_1 \leq t]$$

and

$$P \left[ \frac{X_{n+1} + \cdots + X_{2n}}{\sqrt{n}} \leq t \right] \xrightarrow{D} P[Z_2 \leq t]$$

where $Z_1$ and $Z_2$ are independent, standard normal random variables. Furthermore

$$P \left[ \frac{X_1 + \cdots + X_{2n}}{\sqrt{2n}} \leq t \right] \xrightarrow{D} P[Z_3 \leq t]$$

Since

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} + \frac{X_{n+1} + \cdots + X_{2n}}{\sqrt{n}} = \frac{X_1 + \cdots + X_{2n}}{\sqrt{n}} = \sqrt{2} \frac{X_1 + \cdots + X_{2n}}{\sqrt{2n}}$$

it seems reasonable that the $Z_1 + Z_2$ has the same distribution as $\sqrt{2}Z_3$, that is $Z_1 + Z_2$ is normal with variance 2.

Sources

This section is adapted from: Eisenberg and Sullivan [2] and Eisenberg [1].
Problems to Work for Understanding

1. Cite a reference that demonstrates that the distribution of the sum of independent random variables is the convolution of the distributions of the two independent random variables.

2. Show by direct computation of the convolution of the distributions that the distribution of the sum of independent normal random variables is again normal.

3. Show that $X$ is normal with mean $a$ and variance $b$ if and only if it can be written as $X = a + bZ$ where $Z$ is standard normal with mean 0 and variance 1.

4. Suppose that the joint random variables $(X, Y)$ are uniformly distributed over the unit disk. Show that $X$ has density $f_X(x) = \frac{2}{\pi} \sqrt{1 - x^2}$ for $-1 \leq x \leq 1$. Using the ideas from the rotation proof, show that $aX + bY$ has density $f_c(x) = \frac{2}{c\pi} \sqrt{1 - \frac{x^2}{c^2}}$ for $-c \leq x \leq c$ where $c = \sqrt{a^2 + b^2}$. 
Reading Suggestion:

References


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Outside Readings and Links:

1. Transformations of Multiple Random Variables, Sum of Two Random Variables

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