Starting with

\[ V_1(j) = P\{X_1 = j, S_1 = s_1\} = p_j p(s_1|j) \]

we now use the recursive identity (4.37) to determine \( V_2(j) \) for each \( j \); then \( V_3(j) \) for each \( j \); and so on, up to \( V_n(j) \) for each \( j \).

To obtain the maximizing sequence of states, we work in the reverse direction. Let \( j_n \) be the value (or any of the values if there are more than one) of \( j \) that maximizes \( V_n(j) \). Thus \( j_n \) is the final state of a maximizing state sequence. Also, for \( k < n \), let \( i_k(j) \) be a value of \( i \) that maximizes \( P_{i,j_k} V_k(i) \). Then

\[
\begin{align*}
\max_{i_1,\ldots,i_n} P\{X_n = (i_1,\ldots,i_n), S^n = s_n\} \\
= \max_j V_n(j) \\
= V_n(j_n) \\
= \max_{i_1,\ldots,i_{n-1}} P\{X_n = (i_1,\ldots,i_{n-1}, j_n), S^n = s_n\} \\
= p(s_n|j_n) \max_i P_{i,j_n} V_{n-1}(i) \\
= p(s_n|j_n) P_{i_{n-1}(j_n), j_n} V_{n-1}(i_{n-1}(j_n))
\end{align*}
\]

Thus, \( i_{n-1}(j_n) \) is the next to last state of the maximizing sequence. Continuing in this manner, the second from the last state of the maximizing sequence is \( i_{n-2}(i_{n-1}(j_n)) \), and so on.

The preceding approach to finding the most likely sequence of states given a prescribed sequence of signals is known as the \textit{Viterbi Algorithm}.

\textbf{Exercises}

*1. Three white and three black balls are distributed in two urns in such a way that each contains three balls. We say that the system is in state \( i, i = 0, 1, 2, 3 \), if the first urn contains \( i \) white balls. At each step, we draw one ball from each urn and place the ball drawn from the first urn into the second, and conversely with the ball from the second urn. Let \( X_n \) denote the state of the system after the \( n \)th step. Explain why \( \{X_n, n = 0, 1, 2, \ldots\} \) is a Markov chain and calculate its transition probability matrix.

2. Suppose that whether or not it rains today depends on previous weather conditions through the last three days. Show how this system may be analyzed by using a Markov chain. How many states are needed?
3. In Exercise 2, suppose that if it has rained for the past three days, then it will rain today with probability 0.8; if it did not rain for any of the past three days, then it will rain today with probability 0.2; and in any other case the weather today will, with probability 0.6, be the same as the weather yesterday. Determine P for this Markov chain.

*4. Consider a process \{X_n, n = 0, 1, \ldots\} which takes on the values 0, 1, or 2. Suppose

\[
P\{X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0\} = \begin{cases} p_{ij}, & \text{when } n \text{ is even} \\ p_{ij}^{\text{II}}, & \text{when } n \text{ is odd} \end{cases}
\]

where \(\sum_{j=0}^{2} p_{ij} = \sum_{j=0}^{2} p_{ij}^{\text{II}} = 1, i = 0, 1, 2\). Is \{X_n, n \geq 0\} a Markov chain? If not, then show how, by enlarging the state space, we may transform it into a Markov chain.

5. A Markov chain \{X_n, n \geq 0\} with states 0, 1, 2, has the transition probability matrix

\[
\begin{bmatrix}
\frac{1}{2} & \frac{1}{3} & \frac{1}{6} \\
0 & \frac{1}{3} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2}
\end{bmatrix}
\]

If \(P\{X_0 = 0\} = P\{X_0 = 1\} = \frac{1}{4}\), find \(E[X_3]\).

6. Let the transition probability matrix of a two-state Markov chain be given, as in Example 4.2, by

\[
P = \begin{bmatrix}
p & 1-p \\ 1-p & p
\end{bmatrix}
\]

Show by mathematical induction that

\[
p^{(n)} = \begin{bmatrix}
\frac{1}{2} + \frac{1}{2}(2p - 1)^n & \frac{1}{2} - \frac{1}{2}(2p - 1)^n \\
\frac{1}{2} - \frac{1}{2}(2p - 1)^n & \frac{1}{2} + \frac{1}{2}(2p - 1)^n
\end{bmatrix}
\]

7. In Example 4.4 suppose that it has rained neither yesterday nor the day before yesterday. What is the probability that it will rain tomorrow?

8. Suppose that coin 1 has probability 0.7 of coming up heads, and coin 2 has probability 0.6 of coming up heads. If the coin flipped today comes up heads, then we select coin 1 to flip tomorrow, and if it comes up tails, then we select coin 2 to flip tomorrow. If the coin initially flipped is equally likely to be coin 1 or coin 2, then what is the probability that the coin flipped on the third day after the initial flip is coin 1?
9. Suppose in Exercise 8 that the coin flipped on Monday comes up heads. What is the probability that the coin flipped on Friday of the same week also comes up heads?

10. In Example 4.3, Gary is currently in a cheerful mood. What is the probability that he is not in a glum mood on any of the following three days?

11. In Example 4.3, Gary was in a glum mood four days ago. Given that he hasn’t felt cheerful in a week, what is the probability he is feeling glum today?

12. For a Markov chain \( \{X_n, n \geq 0\} \) with transition probabilities \( P_{i,j} \), consider the conditional probability that \( X_n = m \) given that the chain started at time 0 in state \( i \) and has not yet entered state \( r \) by time \( n \), where \( r \) is a specified state not equal to either \( i \) or \( m \). We are interested in whether this conditional probability is equal to the \( n \) stage transition probability of a Markov chain whose state space does not include state \( r \) and whose transition probabilities are

\[
Q_{i,j} = \frac{P_{i,j}}{1 - P_{i,r}}, \quad i, j \neq r
\]

Either prove the equality

\[
P\{X_n = m | X_0 = i, X_k \neq r, k = 1, \ldots, n\} = Q^n_{i,m}
\]

or construct a counterexample.

13. Let \( P \) be the transition probability matrix of a Markov chain. Argue that if for some positive integer \( r \), \( P^r \) has all positive entries, then so does \( P^n \), for all integers \( n \geq r \).

14. Specify the classes of the following Markov chains, and determine whether they are transient or recurrent:

\[
P_1 = \begin{bmatrix}
0 & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & 0 & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & 0
\end{bmatrix}, \quad P_2 = \begin{bmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 \\
0 & 0 & 1 & 0
\end{bmatrix},
\]

\[
P_3 = \begin{bmatrix}
\frac{1}{3} & 0 & \frac{1}{2} & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 \\
\frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & \frac{1}{2} & \frac{1}{2}
\end{bmatrix}, \quad P_4 = \begin{bmatrix}
\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\
\frac{1}{4} & \frac{3}{4} & 0 & 0 & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

15. Prove that if the number of states in a Markov chain is \( M \), and if state \( j \) can be reached from state \( i \), then it can be reached in \( M \) steps or less.
*16. Show that if state \( i \) is recurrent and state \( j \) does not communicate with state \( i \), then \( P_{ij} = 0 \). This implies that once a process enters a recurrent class of states it can never leave that class. For this reason, a recurrent class is often referred to as a closed class.

17. For the random walk of Example 4.15 use the strong law of large numbers to give another proof that the Markov chain is transient when \( p \neq \frac{1}{2} \).

**Hint:** Note that the state at time \( n \) can be written as \( \sum_{i=1}^{n} Y_i \) where the \( Y_i \)'s are independent and \( P\{Y_i = 1\} = p = 1 - P\{Y_i = -1\} \). Argue that if \( p > \frac{1}{2} \), then, by the strong law of large numbers, \( \sum_{i=1}^{n} Y_i \to \infty \) as \( n \to \infty \) and hence the initial state 0 can be visited only finitely often, and hence must be transient. A similar argument holds when \( p < \frac{1}{2} \).

18. Coin 1 comes up heads with probability 0.6 and coin 2 with probability 0.5. A coin is continually flipped until it comes up tails, at which time that coin is put aside and we start flipping the other one.

(a) What proportion of flips use coin 1?
(b) If we start the process with coin 1 what is the probability that coin 2 is used on the fifth flip?

19. For Example 4.4, calculate the proportion of days that it rains.

20. A transition probability matrix \( P \) is said to be doubly stochastic if the sum over each column equals one; that is,
\[
\sum_i P_{ij} = 1, \quad \text{for all } j
\]
If such a chain is irreducible and aperiodic and consists of \( M + 1 \) states \( 0, 1, \ldots, M \), show that the limiting probabilities are given by
\[
\pi_j = \frac{1}{M+1}, \quad j = 0, 1, \ldots, M
\]

*21. A DNA nucleotide has any of 4 values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability \( 1 - 3\alpha \), and if it does change then it is equally likely to change to any of the other 3 values, for some \( 0 < \alpha < \frac{1}{3} \).

(a) Show that \( P_{1,1}^n = \frac{1}{4} + \frac{3}{4}(1 - 4\alpha)^n \).
(b) What is the long run proportion of time the chain is in each state?

22. Let \( Y_n \) be the sum of \( n \) independent rolls of a fair die. Find
\[
\lim_{n \to \infty} P\{Y_n \text{ is a multiple of } 13\}
**Hint:** Define an appropriate Markov chain and apply the results of Exercise 20.

23. Trials are performed in sequence. If the last two trials were successes, then the next trial is a success with probability 0.8; otherwise the next trial is a success with probability 0.5. In the long run, what proportion of trials are successes?

24. Consider three urns, one colored red, one white, and one blue. The red urn contains 1 red and 4 blue balls; the white urn contains 3 white balls, 2 red balls, and 2 blue balls; the blue urn contains 4 white balls, 3 red balls, and 2 blue balls. At the initial stage, a ball is randomly selected from the red urn and then returned to that urn. At every subsequent stage, a ball is randomly selected from the urn whose color is the same as that of the ball previously selected and is then returned to that urn. In the long run, what proportion of the selected balls are red? What proportion are white? What proportion are blue?

25. Each morning an individual leaves his house and goes for a run. He is equally likely to leave either from his front or back door. Upon leaving the house, he chooses a pair of running shoes (or goes running barefoot if there are no shoes at the door from which he departed). On his return he is equally likely to enter, and leave his running shoes, either by the front or back door. If he owns a total of $k$ pairs of running shoes, what proportion of the time does he run barefooted?

26. Consider the following approach to shuffling a deck of $n$ cards. Starting with any initial ordering of the cards, one of the numbers 1, 2, ..., $n$ is randomly chosen in such a manner that each one is equally likely to be selected. If number $i$ is chosen, then we take the card that is in position $i$ and put it on top of the deck—that is, we put that card in position 1. We then repeatedly perform the same operation. Show that, in the limit, the deck is perfectly shuffled in the sense that the resultant ordering is equally likely to be any of the $n!$ possible orderings.

*27. Determine the limiting probabilities $\pi_j$ for the model presented in Exercise 1. Give an intuitive explanation of your answer.

28. For a series of dependent trials the probability of success on any trial is $(k+1)/(k+2)$ where $k$ is equal to the number of successes on the previous two trials. Compute $\lim_{n \to \infty} P\{\text{success on the } n\text{th trial}\}$.

29. An organization has $N$ employees where $N$ is a large number. Each employee has one of three possible job classifications and changes classifications (independently) according to a Markov chain with transition probabilities

$$
\begin{bmatrix}
0.7 & 0.2 & 0.1 \\
0.2 & 0.6 & 0.2 \\
0.1 & 0.4 & 0.5 \\
\end{bmatrix}
$$

What percentage of employees are in each classification?
30. Three out of every four trucks on the road are followed by a car, while only one out of every five cars is followed by a truck. What fraction of vehicles on the road are trucks?

31. A certain town never has two sunny days in a row. Each day is classified as being either sunny, cloudy (but dry), or rainy. If it is sunny one day, then it is equally likely to be either cloudy or rainy the next day. If it is rainy or cloudy one day, then there is one chance in two that it will be the same the next day, and if it changes then it is equally likely to be either of the other two possibilities. In the long run, what proportion of days are sunny? What proportion are cloudy?

*32. Each of two switches is either on or off during a day. On day \( n \), each switch will independently be on with probability

\[
\frac{1 + \text{number of on switches during day } n - 1}{4}
\]

For instance, if both switches are on during day \( n - 1 \), then each will independently be on during day \( n \) with probability \( \frac{3}{4} \). What fraction of days are both switches on? What fraction are both off?

33. A professor continually gives exams to her students. She can give three possible types of exams, and her class is graded as either having done well or badly. Let \( p_i \) denote the probability that the class does well on a type \( i \) exam, and suppose that \( p_1 = 0.3 \), \( p_2 = 0.6 \), and \( p_3 = 0.9 \). If the class does well on an exam, then the next exam is equally likely to be any of the three types. If the class does badly, then the next exam is always type 1. What proportion of exams are type \( i \), \( i = 1, 2, 3 \)?

34. A flea moves around the vertices of a triangle in the following manner: Whenever it is at vertex \( i \) it moves to its clockwise neighbor vertex with probability \( p_i \) and to the counterclockwise neighbor with probability \( q_i = 1 - p_i \), \( i = 1, 2, 3 \).

(a) Find the proportion of time that the flea is at each of the vertices.

(b) How often does the flea make a counterclockwise move which is then followed by five consecutive clockwise moves?

35. Consider a Markov chain with states 0, 1, 2, 3, 4. Suppose \( P_{0,4} = 1 \); and suppose that when the chain is in state \( i \), \( i > 0 \), the next state is equally likely to be any of the states 0, 1, \ldots, \( i - 1 \). Find the limiting probabilities of this Markov chain.

36. The state of a process changes daily according to a two-state Markov chain. If the process is in state \( i \) during one day, then it is in state \( j \) the follow-
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ing day with probability \( P_{i,j} \), where

\[
P_{0,0} = 0.4, \quad P_{0,1} = 0.6, \quad P_{1,0} = 0.2, \quad P_{1,1} = 0.8
\]

Every day a message is sent. If the state of the Markov chain that day is \( i \) then the message sent is “good” with probability \( p_i \) and is “bad” with probability \( q_i = 1 - p_i \), \( i = 0, 1 \)

(a) If the process is in state 0 on Monday, what is the probability that a good message is sent on Tuesday?
(b) If the process is in state 0 on Monday, what is the probability that a good message is sent on Friday?
(c) In the long run, what proportion of messages are good?
(d) Let \( Y_n \) equal 1 if a good message is sent on day \( n \) and let it equal 2 otherwise. Is \( \{Y_n, n \geq 1\} \) a Markov chain? If so, give its transition probability matrix.
If not, briefly explain why not.

37. Show that the stationary probabilities for the Markov chain having transition probabilities \( P_{i,j} \) are also the stationary probabilities for the Markov chain whose transition probabilities \( Q_{i,j} \) are given by

\[
Q_{i,j} = P_{i,j}^k
\]

for any specified positive integer \( k \).

38. Recall that state \( i \) is said to be positive recurrent if \( m_{i,i} < \infty \), where \( m_{i,i} \) is the expected number of transitions until the Markov chain, starting in state \( i \), makes a transition back into that state. Because \( \pi_i \), the long run proportion of time the Markov chain, starting in state \( i \), spends in state \( i \), satisfies

\[
\pi_i = \frac{1}{m_{i,i}}
\]

it follows that state \( i \) is positive recurrent if and only if \( \pi_i > 0 \). Suppose that state \( i \) is positive recurrent and that state \( i \) communicates with state \( j \). Show that state \( j \) is also positive recurrent by arguing that there is an integer \( n \) such that

\[
\pi_j \geq \pi_i P_{i,j}^n > 0
\]

39. Recall that a recurrent state that is not positive recurrent is called null recurrent. Use the result of Exercise 38 to prove that null recurrence is a class property. That is, if state \( i \) is null recurrent and state \( i \) communicates with state \( j \), show that state \( j \) is also null recurrent.
40. It follows from the argument made in Exercise 38 that state $i$ is null recurrent if it is recurrent and $\pi_i = 0$. Consider the one-dimensional symmetric random walk of Example 4.15.

(a) Argue that $\pi_i = \pi_0$ for all $i$.
(b) Argue that all states are null recurrent.

*41. Let $\pi_i$ denote the long-run proportion of time a given irreducible Markov chain is in state $i$.

(a) Explain why $\pi_i$ is also the proportion of transitions that are into state $i$ as well as being the proportion of transitions that are from state $i$.
(b) $\pi_i P_{ij}$ represents the proportion of transitions that satisfy what property?
(c) $\sum_i \pi_i P_{ij}$ represent the proportion of transitions that satisfy what property?
(d) Using the preceding explain why

$$\pi_j = \sum_i \pi_i P_{ij}$$

42. Let $A$ be a set of states, and let $A^c$ be the remaining states.

(a) What is the interpretation of

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij}?$$

(b) What is the interpretation of

$$\sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}?$$

(c) Explain the identity

$$\sum_{i \in A} \sum_{j \in A^c} \pi_i P_{ij} = \sum_{i \in A^c} \sum_{j \in A} \pi_i P_{ij}$$

43. Each day, one of $n$ possible elements is requested, the $i$th one with probability $P_i, i \geq 1, \sum_i^n P_i = 1$. These elements are at all times arranged in an ordered list which is revised as follows: The element selected is moved to the front of the list with the relative positions of all the other elements remaining unchanged. Define the state at any time to be the list ordering at that time and note that there are $n!$ possible states.

(a) Argue that the preceding is a Markov chain.
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(b) For any state \(i_1, \ldots, i_n\) (which is a permutation of \(1, 2, \ldots, n\)), let \(\pi(i_1, \ldots, i_n)\) denote the limiting probability. In order for the state to be \(i_1, \ldots, i_n\), it is necessary for the last request to be for \(i_1\), the last non-\(i_1\) request for \(i_2\), the last non-\(i_1\) or \(i_2\) request for \(i_3\), and so on. Hence, it appears intuitive that

\[
\pi(i_1, \ldots, i_n) = P_{i_1} P_{i_2} \frac{P_{i_3}}{1 - P_{i_1}} \frac{P_{i_4}}{1 - P_{i_1} - P_{i_2}} \cdots \frac{P_{i_n}}{1 - P_{i_1} - \cdots - P_{i_{n-1}}}
\]

Verify when \(n = 3\) that the preceding are indeed the limiting probabilities.

44. Suppose that a population consists of a fixed number, say, \(m\), of genes in any generation. Each gene is one of two possible genetic types. If any generation has exactly \(i\) (of its \(m\)) genes being type 1, then the next generation will have \(j\) type 1 (and \(m - j\) type 2) genes with probability

\[
\binom{m}{j} \left( \frac{i}{m} \right)^j \left( \frac{m-i}{m} \right)^{m-j}, \quad j = 0, 1, \ldots, m
\]

Let \(X_n\) denote the number of type 1 genes in the \(n\)th generation, and assume that \(X_0 = i\).

(a) Find \(E[X_n]\).

(b) What is the probability that eventually all the genes will be type 1?

45. Consider an irreducible finite Markov chain with states \(0, 1, \ldots, N\).

(a) Starting in state \(i\), what is the probability the process will ever visit state \(j\)? Explain!

(b) Let \(x_i = P\{\text{visit state } N \text{ before state } 0|\text{start in } i\}\). Compute a set of linear equations which the \(x_i\) satisfy, \(i = 0, 1, \ldots, N\).

(c) If \(\sum_j P_{ij} = i\) for \(i = 1, \ldots, N - 1\), show that \(x_i = i/N\) is a solution to the equations in part (b)

46. An individual possesses \(r\) umbrellas which he employs in going from his home to office, and vice versa. If he is at home (the office) at the beginning (end) of a day and it is raining, then he will take an umbrella with him to the office (home), provided there is one to be taken. If it is not raining, then he never takes an umbrella. Assume that, independent of the past, it rains at the beginning (end) of a day with probability \(p\).

(i) Define a Markov chain with \(r + 1\) states which will help us to determine the proportion of time that our man gets wet. (Note: He gets wet if it is raining, and all umbrellas are at his other location.)
(ii) Show that the limiting probabilities are given by
\[ \pi_i = \begin{cases} \frac{q}{r+q}, & \text{if } i = 0 \\ \frac{1}{r+q}, & \text{if } i = 1, \ldots, r \end{cases} \]
where \( q = 1 - p \)

(iii) What fraction of time does our man get wet?

(iv) When \( r = 3 \), what value of \( p \) maximizes the fraction of time he gets wet

*47. Let \( \{X_n, n \geq 0\} \) denote an ergodic Markov chain with limiting probabilities \( \pi_i \). Define the process \( \{Y_n, n \geq 1\} \) by \( Y_n = (X_{n-1}, X_n) \). That is, \( Y_n \) keeps track of the last two states of the original chain. Is \( \{Y_n, n \geq 1\} \) a Markov chain? If so, determine its transition probabilities and find
\[ \lim_{n \to \infty} P(Y_n = (i, j)) \]

48. Verify the transition probability matrix given in Example 4.20.

49. Let \( P^{(1)} \) and \( P^{(2)} \) denote transition probability matrices for ergodic Markov chains having the same state space. Let \( \pi^1 \) and \( \pi^2 \) denote the stationary (limiting) probability vectors for the two chains. Consider a process defined as follows:

(i) \( X_0 = 1 \). A coin is then flipped and if it comes up heads, then the remaining states \( X_1, \ldots \) are obtained from the transition probability matrix \( P^{(1)} \) and if tails from the matrix \( P^{(2)} \). Is \( \{X_n, n \geq 0\} \) a Markov chain? If \( p = P\{\text{coin comes up heads}\} \), what is \( \lim_{n \to \infty} P(X_n = i) \)?

(ii) \( X_0 = 1 \). At each stage the coin is flipped and if it comes up heads, then the next state is chosen according to \( P^{(1)} \) and if tails comes up, then it is chosen according to \( P^{(2)} \). In this case do the successive states constitute a Markov chain? If so, determine the transition probabilities. Show by a counterexample that the limiting probabilities are not the same as in part (i).

50. In Exercise 8, if today’s flip lands heads, what is the expected number of additional flips needed until the pattern \( t, t, h, t, h, t, t \) occurs?

51. In Example 4.3, Gary is in a cheerful mood today. Find the expected number of days until he has been glum for three consecutive days.

52. A taxi driver provides service in two zones of a city. Fares picked up in zone \( A \) will have destinations in zone \( A \) with probability 0.6 or in zone \( B \) with probability 0.4. Fares picked up in zone \( B \) will have destinations in zone \( A \) with probability 0.3 or in zone \( B \) with probability 0.7. The driver’s expected profit for a trip entirely in zone \( A \) is 6; for a trip entirely in zone \( B \) is 8; and for a trip that involves both zones is 12. Find the taxi driver’s average profit per trip.
53. Find the average premium received per policyholder of the insurance company of Example 4.23 if \( \lambda = 1/4 \) for one-third of its clients, and \( \lambda = 1/2 \) for two-thirds of its clients.

54. Consider the Ehrenfest urn model in which \( M \) molecules are distributed between two urns, and at each time point one of the molecules is chosen at random and is then removed from its urn and placed in the other one. Let \( X_n \) denote the number of molecules in urn 1 after the \( n \)th switch and let \( \mu_n = E[X_n] \). Show that

(i) \( \mu_{n+1} = 1 + (1 - 2/M)\mu_n \).

(ii) Use (i) to prove that

\[
\mu_n = \frac{M}{2} + \left( \frac{M - 2}{M} \right)^n \left( E[X_0] - \frac{M}{2} \right)
\]

55. Consider a population of individuals each of whom possesses two genes which can be either type \( A \) or type \( a \). Suppose that in outward appearance type \( A \) is dominant and type \( a \) is recessive. (That is, an individual will have only the outward characteristics of the recessive gene if its pair is \( aa \).) Suppose that the population has stabilized, and the percentages of individuals having respective gene pairs \( AA, aa, \) and \( Aa \) are \( p, q, \) and \( r \). Call an individual dominant or recessive depending on the outward characteristics it exhibits. Let \( S_{11} \) denote the probability that an offspring of two dominant parents will be recessive; and let \( S_{10} \) denote the probability that the offspring of one dominant and one recessive parent will be recessive. Compute \( S_{11} \) and \( S_{10} \) to show that \( S_{11} = S_{10}^2 \).

(The quantities \( S_{10} \) and \( S_{11} \) are known in the genetics literature as Snyder’s ratios.)

56. Suppose that on each play of the game a gambler either wins 1 with probability \( p \) or loses 1 with probability \( 1 - p \). The gambler continues betting until she or he is either winning \( n \) or losing \( m \). What is the probability that the gambler quits a winner?

57. A particle moves among \( n + 1 \) vertices that are situated on a circle in the following manner. At each step it moves one step either in the clockwise direction with probability \( p \) or the counterclockwise direction with probability \( q = 1 - p \). Starting at a specified state, call it state 0, let \( T \) be the time of the first return to state 0. Find the probability that all states have been visited by time \( T \).

**Hint:** Condition on the initial transition and then use results from the gambler’s ruin problem.
58. In the gambler’s ruin problem of Section 4.5.1, suppose the gambler’s fortune is presently $i$, and suppose that we know that the gambler’s fortune will eventually reach $N$ (before it goes to 0). Given this information, show that the probability he wins the next gamble is

$$\frac{p[1-(q/p)^{i+1}]}{1-(q/p)^i}, \quad \text{if } p \neq \frac{1}{2}$$

$$\frac{i + 1}{2i}, \quad \text{if } p = \frac{1}{2}$$

Hint: The probability we want is

$$P\{X_{n+1} = i + 1|X_n = i, \lim_{m \to \infty} X_m = N\}$$

$$= \frac{P\{X_{n+1} = i + 1, \lim_{m \to \infty} X_m = N|X_n = i\}}{P|\lim_{m \to \infty} X_m = N|X_n = i\}$$

59. For the gambler’s ruin model of Section 4.5.1, let $M_i$ denote the mean number of games that must be played until the gambler either goes broke or reaches a fortune of $N$, given that he starts with $i, i = 0, 1, \ldots, N$. Show that $M_i$ satisfies

$$M_0 = M_N = 0; \quad M_i = 1 + pM_{i+1} + qM_{i-1}, \quad i = 1, \ldots, N - 1$$

60. Solve the equations given in Exercise 59 to obtain

$$M_i = i(N - i), \quad \text{if } p = \frac{1}{2}$$

$$= \frac{i}{q - p} \left(\frac{N}{q - p} \frac{1 - (q/p)^i}{1 - (q/p)^N}\right), \quad \text{if } p \neq \frac{1}{2}$$

61. Suppose in the gambler’s ruin problem that the probability of winning a bet depends on the gambler’s present fortune. Specifically, suppose that $\alpha_i$ is the probability that the gambler wins a bet when his or her fortune is $i$. Given that the gambler’s initial fortune is $i$, let $P(i)$ denote the probability that the gambler’s fortune reaches $N$ before 0.

(a) Derive a formula that relates $P(i)$ to $P(i - 1)$ and $P(i + 1)$.

(b) Using the same approach as in the gambler’s ruin problem, solve the equation of part (a) for $P(i)$.

(c) Suppose that $i$ balls are initially in urn 1 and $N - i$ are in urn 2, and suppose that at each stage one of the $N$ balls is randomly chosen, taken from whichever urn it is in, and placed in the other urn. Find the probability that the first urn becomes empty before the second.
*62. In Exercise 21,
   (a) what is the expected number of steps the particle takes to return to the
   starting position?
   (b) what is the probability that all other positions are visited before the particle
   returns to its starting state?

63. For the Markov chain with states 1, 2, 3, 4 whose transition probability
   matrix $P$ is as specified below find $f_{i3}$ and $s_{i3}$ for $i = 1, 2, 3.$
   
   $P = \begin{bmatrix}
   0.4 & 0.2 & 0.1 & 0.3 \\
   0.1 & 0.5 & 0.2 & 0.2 \\
   0.3 & 0.4 & 0.2 & 0.1 \\
   0 & 0 & 0 & 1 
   \end{bmatrix}$

64. Consider a branching process having $\mu < 1.$ Show that if $X_0 = 1,$ then
   the expected number of individuals that ever exist in this population is given by
   $1/(1 - \mu).$ What if $X_0 = n$?

65. In a branching process having $X_0 = 1$ and $\mu > 1,$ prove that $\pi_0$ is the smallest
   positive number satisfying Equation (4.16).
   
   Hint: Let $\pi$ be any solution of $\pi = \sum_{j=0}^{\infty} \pi_j P_j.$ Show by mathematical induction that $\pi \geq P\{X_n = 0\}$ for all $n,$ and let $n \to \infty.$ In using the induction argue that
   
   $P\{X_n = 0\} = \sum_{j=0}^{\infty} (P\{X_{n-1} = 0\})^j P_j$

66. For a branching process, calculate $\pi_0$ when
   (a) $P_0 = \frac{1}{4}, P_2 = \frac{3}{4}.$
   (b) $P_0 = \frac{1}{4}, P_1 = \frac{1}{2}, P_2 = \frac{1}{4}.$
   (c) $P_0 = \frac{1}{6}, P_1 = \frac{1}{2}, P_3 = \frac{1}{3}.$

67. At all times, an urn contains $N$ balls—some white balls and some black
   balls. At each stage, a coin having probability $p, 0 < p < 1,$ of landing heads
   is flipped. If heads appears, then a ball is chosen at random from the urn and is
   replaced by a white ball; if tails appears, then a ball is chosen from the urn and is
   replaced by a black ball. Let $X_n$ denote the number of white balls in the urn after
   the $n$th stage.
   
   (a) Is $\{X_n, n \geq 0\}$ a Markov chain? If so, explain why.
   (b) What are its classes? What are their periods? Are they transient or recurrent?
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(c) Compute the transition probabilities \( P_{ij} \).
(d) Let \( N = 2 \). Find the proportion of time in each state.
(e) Based on your answer in part (d) and your intuition, guess the answer for the limiting probability in the general case.
(f) Prove your guess in part (e) either by showing that Equation (4.7) is satisfied or by using the results of Example 4.31.
(g) If \( p = 1 \), what is the expected time until there are only white balls in the urn if initially there are \( i \) white and \( N - i \) black?

*68.  (a) Show that the limiting probabilities of the reversed Markov chain are the same as for the forward chain by showing that they satisfy the equations

\[ \pi_j = \sum_i \pi_i Q_{ij} \]

(b) Give an intuitive explanation for the result of part (a).

69.  \( M \) balls are initially distributed among \( m \) urns. At each stage one of the balls is selected at random, taken from whichever urn it is in, and then placed, at random, in one of the other \( M - 1 \) urns. Consider the Markov chain whose state at any time is the vector \((n_1, \ldots, n_m)\) where \( n_i \) denotes the number of balls in urn \( i \). Guess at the limiting probabilities for this Markov chain and then verify your guess and show at the same time that the Markov chain is time reversible.

70.  A total of \( m \) white and \( m \) black balls are distributed among two urns, with each urn containing \( m \) balls. At each stage, a ball is randomly selected from each urn and the two selected balls are interchanged. Let \( X_n \) denote the number of black balls in urn 1 after the \( n \)th interchange.

(a) Give the transition probabilities of the Markov chain \( X_n, n \geq 0 \).
(b) Without any computations, what do you think are the limiting probabilities of this chain?
(c) Find the limiting probabilities and show that the stationary chain is time reversible.

71.  It follows from Theorem 4.2 that for a time reversible Markov chain

\[ P_{ij} P_{jk} P_{ki} = P_{ik} P_{kj} P_{ji}, \quad \text{for all } i, j, k \]

It turns out that if the state space is finite and \( P_{ij} > 0 \) for all \( i, j \), then the preceding is also a sufficient condition for time reversibility. [That is, in this case, we need only check Equation (4.26) for paths from \( i \) to \( i \) that have only two intermediate states.] Prove this.
**Hint:** Fix \( i \) and show that the equations

\[
\pi_j P_{jk} = \pi_k P_{kj}
\]

are satisfied by \( \pi_j = c P_{ij}/P_{ji} \), where \( c \) is chosen so that \( \sum_j \pi_j = 1 \).

**72.** For a time reversible Markov chain, argue that the rate at which transitions from \( i \) to \( j \) to \( k \) occur must equal the rate at which transitions from \( k \) to \( j \) to \( i \) occur.

**73.** Show that the Markov chain of Exercise 31 is time reversible.

**74.** A group of \( n \) processors is arranged in an ordered list. When a job arrives, the first processor in line attempts it; if it is unsuccessful, then the next in line tries it; if it too is unsuccessful, then the next in line tries it, and so on. When the job is successfully processed or after all processors have been unsuccessful, the job leaves the system. At this point we are allowed to reorder the processors, and a new job appears. Suppose that we use the one-closer reordering rule, which moves the processor that was successful one closer to the front of the line by interchanging its position with the one in front of it. If all processors were unsuccessful (or if the processor in the first position was successful), then the ordering remains the same. Suppose that each time processor \( i \) attempts a job then, independently of anything else, it is successful with probability \( p_i \).

(a) Define an appropriate Markov chain to analyze this model.
(b) Show that this Markov chain is time reversible.
(c) Find the long-run probabilities.

**75.** A Markov chain is said to be a tree process if

(i) \( P_{ij} > 0 \) whenever \( P_{ji} > 0 \),

(ii) for every pair of states \( i \) and \( j, i \neq j \), there is a unique sequence of distinct states \( i = i_0, i_1, \ldots, i_{n-1}, i_n = j \) such that

\[
P_{i_k,i_{k+1}} > 0, \quad k = 0, 1, \ldots, n - 1
\]

In other words, a Markov chain is a tree process if for every pair of distinct states \( i \) and \( j \) there is a unique way for the process to go from \( i \) to \( j \) without reentering a state (and this path is the reverse of the unique path from \( j \) to \( i \)). Argue that an ergodic tree process is time reversible.

**76.** On a chessboard compute the expected number of plays it takes a knight, starting in one of the four corners of the chessboard, to return to its initial position if we assume that at each play it is equally likely to choose any of its legal moves. (No other pieces are on the board.)
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Hint: Make use of Example 4.32.

77. In a Markov decision problem, another criterion often used, different than the expected average return per unit time, is that of the expected discounted return. In this criterion we choose a number \( \alpha, 0 < \alpha < 1 \), and try to choose a policy so as to maximize \( E[\sum_{i=0}^{\infty} \alpha^i R(X_i, a_i)] \) (that is, rewards at time \( n \) are discounted at rate \( \alpha^n \)). Suppose that the initial state is chosen according to the probabilities \( b_i \). That is,

\[
P\{X_0 = i\} = b_i, \quad i = 1, \ldots, n
\]

For a given policy \( \beta \) let \( y_{ja} \) denote the expected discounted time that the process is in state \( j \) and action \( a \) is chosen. That is,

\[
y_{ja} = E_\beta \left[ \sum_{n=0}^{\infty} \alpha^n I\{X_n = j, a_n = a\} \right]
\]

where for any event \( A \) the indicator variable \( I_A \) is defined by

\[
I_A = \begin{cases} 
1, & \text{if } A \text{ occurs} \\
0, & \text{otherwise}
\end{cases}
\]

(a) Show that

\[
\sum_a y_{ja} = E_\beta \left[ \sum_{n=0}^{\infty} \alpha^n I\{X_n = j\} \right]
\]

or, in other words, \( \sum_a y_{ja} \) is the expected discounted time in state \( j \) under \( \beta \).

(b) Show that

\[
\sum_j \sum_a y_{ja} = \frac{1}{1 - \alpha},
\]

\[
\sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a)
\]

Hint: For the second equation, use the identity

\[
I\{X_{n+1} = j\} = \sum_i \sum_a I\{X_{n+1} = j, a_{n+1} = a\} I\{X_n = i, a_n = a\}
\]

Take expectations of the preceding to obtain

\[
E[I_{X_{n+1} = j}] = \sum_i \sum_a E[I_{X_{n+1} = j, a_{n+1} = a}] P_{ij}(a)
\]
(c) Let \( \{y_{ja}\} \) be a set of numbers satisfying
\[
\sum_j \sum_a y_{ja} = \frac{1}{1 - \alpha},
\]
\[
\sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a)
\]
(4.38)

Argue that \( y_{ja} \) can be interpreted as the expected discounted time that the process is in state \( j \) and action \( a \) is chosen when the initial state is chosen according to the probabilities \( b_j \) and the policy \( \beta \), given by
\[
\beta_i(a) = \frac{y_{ia}}{\sum_a y_{ia}}
\]
is employed.

**Hint:** Derive a set of equations for the expected discounted times when policy \( \beta \) is used and show that they are equivalent to Equation (4.38).

(d) Argue that an optimal policy with respect to the expected discounted return criterion can be obtained by first solving the linear program

\[
\text{maximize } \sum_j \sum_a y_{ja} R(j,a),
\]

such that
\[
\sum_j \sum_a y_{ja} = \frac{1}{1 - \alpha},
\]
\[
\sum_a y_{ja} = b_j + \alpha \sum_i \sum_a y_{ia} P_{ij}(a),
\]
\[
y_{ja} \geq 0, \quad \text{all } j, a;
\]

and then defining the policy \( \beta^* \) by
\[
\beta^*_i(a) = \frac{y^*_{ia}}{\sum_a y^*_{ia}}
\]
where the \( y^*_{ja} \) are the solutions of the linear program.

78. For the Markov chain of Exercise 5, suppose that \( p(s|j) \) is the probability that signal \( s \) is emitted when the underlying Markov chain state is \( j, j = 0, 1, 2. \)

(a) What proportion of emissions are signal \( s \)?
(b) What proportion of those times in which signal \( s \) is emitted is 0 the underlying state?

79. In Example 4.39, what is the probability that the first 4 items produced are all acceptable?