

**Math 489/889**  
**Stochastic Processes and**  
**Advanced Mathematical Finance**  
**Final Exam Solutions**

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Tuesday, December 14, 2010

▼ **1**

“Short Answer”, use a single sentence or “True or False” and if false, give a reason why it is false in a single sentence. (If false, 1 point for the answer, 4 points for the reason.)

▼ **(a)**

Short Answer: Why is Geometric Brownian Motion a better model of the stock market than Brownian motion with drift, where the drift parameter is the rate  $\lambda$  of market growth?

Because Geometric Brownian motion is always positive, while there is a positive probability (albeit small) that Brownian Motion with drift can take on negative values.

▼ **(b)**

True or False: The Black-Scholes pricing equation is based on the model that the underlying stock price follows a Brownian Motion.

False, the underlying stock price is assumed to follow a Geometric Brownian motion, a different (but related) stochastic process.

▼ **(c)**

The Black-Scholes pricing equation values an option by taking the present value

of the expected return on the option.

False, the Black Scholes equation prices an option by finding an equivalent dynamic portfolio in the underlying security and a bond and then uses the principle of no-arbitrage to find the equivalent price.

**(d)**

True or False: The closed form solution of the partial differential equation that we call the Black-Scholes formula represents the final word in financial theory.

False, it is the starting point and initial ground-breaking idea that stimulated the whole area of mathematical finance.

**(e)**

True or False: The volatility of a stock price can be estimated from the Black-Scholes Formula if the option values are known from the market.

True, that is called the implied volatility.

**(f)**

True or False: European puts cannot be valued by solving the Black-Scholes equation, only European calls can be valued by solving the Black-Scholes equation.

False, they can by solving a different terminal value problem, one that we did not choose to solve, but could have.

**(g)**

Short answer: What mathematical property of the Black-Scholes equation allows you to write the formula for the value of a strap (a portfolio consisting of one put and two calls, all with the same strike price) in terms of the value for a call and a put other solutions?

The Black-Scholes equation is linear, so that the linear combinations of solutions is again a solution.

**2**

What is the price of a European put option on a non-dividend-paying when the stock price is  $\$69$ , the strike price is  $\$70$ , the risk-free interest rate is 5% per year (continuously compounded), the volatility is 35% per year, and the time to

maturity is 6 months.

with(finance);

[amortization, annuity, blackscholes, cashflows, effectiverate, futurevalue, growingannuity, growingperpetuity, levelcoupon, perpetuity, presentvalue, yieldtomaturity] (2.1)

with(Statistics) :

S := 69;

K := 70;

r := 0.05;

σ := 0.35;

Tminust := 0.5;

69

70

0.05

0.35

0.5

(2.2)

callOption := evalf( blackscholes( S, K, r, Tminust, σ ) );

7.129713803

(2.3)

Use Put-Call Parity to determine the putOption value:

putOption := callOption + K·exp(-r·Tminust) - S;

6.40140764

(2.4)

Now value the Put Option directly with the Black-Scholes formula for a Put:

$$d1 := \text{evalf} \left( \frac{\left( \log \left( \frac{S}{K} \right) + \left( r + \frac{\sigma^2}{2} \right) \cdot (Tminust) \right)}{\sigma \cdot \sqrt{Tminust}} \right);$$

0.1666196621

(2.5)

$$d2 := \text{evalf} \left( \frac{\left( \log \left( \frac{S}{K} \right) + \left( r - \frac{\sigma^2}{2} \right) \cdot (Tminust) \right)}{\sigma \cdot \sqrt{Tminust}} \right)$$

-0.08086771132

(2.6)

checkd2 := evalf( d1 - sigma·√Tminust );

-0.0808677113

(2.7)

Z := RandomVariable(Normal(0, 1));

Phid1 := CDF(Z, -d1);

\_R0

0.433834660924894255

(2.8)

Phid2 := CDF(Z, -d2);

0.532226420764111419

(2.9)

$$C := -S \cdot \text{Phid1} + K \cdot \exp(-r \cdot T_{\text{minust}}) \cdot \text{Phid2};$$

6.40140765

(2.10)

### 3

Use the put-call parity relationship to derive the relationship between

(a) The Delta of European call and the Delta of European put. (The Delta of an option is the rate of change of option value with respect to  $S$ .)

(b) The Theta of European call and a European put. (The Theta of an option is the rate of change of option value with respect to  $t$ .)

Show your complete work.

restart;

$$\text{putCallParity} := C(S, t) - P(S, t) = S - K \cdot \exp(-r \cdot (T - t));$$

$$C(S, t) - P(S, t) = S - K e^{-r(T-t)}$$

(3.1)

#### (a)

$$\frac{\partial}{\partial S} (\text{putCallParity});$$

$$\frac{\partial}{\partial S} C(S, t) - \left( \frac{\partial}{\partial S} P(S, t) \right) = 1$$

(3.1.1)

#### (b)

$$\frac{\partial}{\partial t} (\text{putCallParity});$$

$$\frac{\partial}{\partial t} C(S, t) - \left( \frac{\partial}{\partial t} P(S, t) \right) = -K r e^{-r(T-t)}$$

(3.2.1)

### 4

Find a numerical approximation at  $t = 0.2, 0.4, 0.6, 0.8, 1.0$  to the solution of the Stochastic Differential Equation:

$$dX = (1 - X) dt + dW; \quad X(0) = 0.5$$

Remark: With some general parameters, this stochastic differential equation is a model of a "mean-reverting process" called the Ornstein-Uhlenbeck process, a useful model in physics and mathematics.)

Use  $dt = 0.2$ , and  $N = 100$  in the table of net totals of randomly generated coin flips below.

Ornstein-Uhlenbeck Approximation									
	A	B	C	D	E	F	G	H	I
1	"j"	"t_j"	"X_j"	"(1-X_j)"	"(1-X_j)*dt"	"dW"	"(1-X_j)*dt+dW"	"X_{j+1}"	
2	0	0.	0.5	0.5	0.10	0.2	0.30	0.80	
3	1	0.2	0.80	0.20	0.040	-0.8	-0.760	0.040	
4	2	0.4	0.040	0.960	0.1920	-0.4	-0.2080	-0.1680	
5	3	0.6	-0.1680	1.1680	0.23360	0.	0.23360	0.06560	
6	4	0.8	0.06560	0.93440	0.186880	-0.2	-0.013120	0.052480	
7	5	1.0	0.052480						
8									

## 5

A company's cash position, measured in millions of dollars, follows a general Brownian motion with a drift rate of 0.1 per month, and a volatility rate of 0.16 per month. The initial cash position is 2.0 That is, the cash position at time  $t$  follows the SDE

$$\begin{aligned} dX &= 0.1 dt + 0.16 \; dW \\ X(0) &= 2.0 \end{aligned}$$

*Read the problem and the SDE carefully!*

- (a) What are the probability distributions of the cash position after 1 month, 6 months, and 1 year?  
 (b) What are the probabilities of a negative cash position at the end of 6 months and one year?

### (a)

The cash position at 1 month, 6 months, and 1 year are respectively

$$X(1) \sim N(2.1, (0.16)^2)$$

$$X(6) \sim N(2.6, 6 \cdot (0.16)^2)$$

$$X(12) \sim N(3.2, 12 \cdot (0.16)^2)$$

### (b)

*restart;*

*with(Statistics) :*

$Z := \text{RandomVariable}(\text{Normal}(0, 1));$

$\_R$

**(5.2.1)**

$$\Pr[X(1) < 0] \approx \Pr[(Z - 2.1)/0.16 < -2.1/0.16] = 1.1E-39;$$

$$CDF\left(Z, -\frac{2.1}{0.16}\right)$$

$$1.18391379285191753 \cdot 10^{-39} \quad (5.2.2)$$

$$\Pr[X(6) < 0] \approx \Pr[(Z-2.6)/0.96 < -2.6/0.96] = 0.00338;$$

$$CDF\left(Z, -\frac{2.6}{\sqrt{6} \cdot 0.16}\right)$$

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}(1.354166666 \sqrt{6} \sqrt{2}) \quad (5.2.3)$$

at 20 digits →

$$1.633164756 \cdot 10^{-11} \quad (5.2.4)$$

$$\Pr[X(12) < 0] \approx \Pr[(Z-3.2)/1.92 < -3.2/1.92] = 0.04779;$$

$$CDF\left(Z, -\frac{3.2}{\sqrt{12} \cdot 0.16}\right);$$

$$\frac{1}{2} - \frac{1}{2} \operatorname{erf}(1.666666666 \sqrt{3} \sqrt{2}) \quad (5.2.5)$$

at 20 digits →

$$3.88201832220 \cdot 10^{-9} \quad (5.2.6)$$