

Problem	1	2	3	4	Total
Possible	15	20	20	20	75
Points					

1. (15 points) According to the article “Bullion bulls” on page 81 in the October 8, 2009 issue of *The Economist*, gold has risen from about \$510 per ounce in January 2006 to about \$1050 per ounce in October 2009, 46 months later.

- (a) What is the continuously compounded annual rate of increase of the price of gold over this period?
- (b) In October 2009, one can borrow or lend money at 5% interest, again assume it compounded continuously. In view of this, describe a strategy that will make a profit in October 2010, involving borrowing or lending money, assuming that the rate of increase in the price gold stays constant over this time.
- (c) The article suggests that the rate of increase for gold will stay constant. In view of this, what do you expect to happen to interest rates and what principle allows you to conclude that?

(a) The continuously compounded annual rate of increase of gold over this period is the solution r of the equation

$$g(T) = g(0) \exp(rT)$$

where $T = 46/12$, $g(T) = 1050$, $g(0) = 510$, or

$$r = \ln(g(T)/g(0))/T = \ln(1050/510)/(46/12) = 0.1883829697,$$

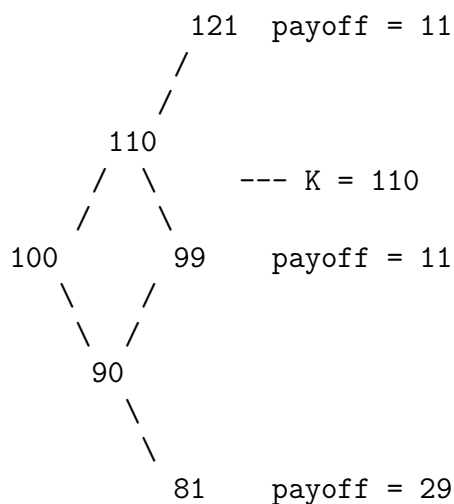
about 18.8% annual rate.

- (b) One could borrow \$1050 at 5% continuously compounded interest and buy one ounce of gold at \$1050. Holding the gold for one year, its value in October 2010 is $1050 \exp(0.188) = 1267.66$. You could sell the gold, making \$217.66 per ounce. You then repay the loan which will require $1050 \cdot \exp(0.05) = 1103.83$, or approximately 53.83 in interest. The profit per ounce of gold is $217.66 - 53.83 = 163.83$.
- (c) The principle of arbitrage says that this opportunity will not stay for long, so if the rate of increase of gold stays constant, then interest rates must rise.

In economic actuality, the situation is not so simple, since credit in October 2009 is not readily available and not many banks would loan for such a venture. Furthermore, the price of gold is also affected by currency values and exchange rates, other commodity prices, and world events. As a result the price of gold is notoriously volatile, so the assumption of a constant rate of increase is optimistic at best.

2. (20 points) A *long straddle option* pays $|S - K|$ if it expires when the underlying stock value is S . The option is a portfolio composed of a call and a put on the same security with K as the strike price for both. A stock currently has price \$100 and goes up or down by 10% in each time period. What is the value of such a long straddle option with strike price $K = 110$ at expiration 2 time units in the future? Assume a *simple* interest rate of 5% in each time period.

Solution: The recombinant binomial tree is:



$$\pi = \frac{1.05 - 0.90}{1.10 - 0.90} = 0.75$$

$$1 - \pi = \frac{1.10 - 1.05}{1.10 - 0.90} = 0.25$$

The value of the option at the time period 1 is either $V_{1,1} = (1/1.05)[0.75 \cdot 11 + 0.25 \cdot 11] = 10.47619048$ or $V_{1,0} = (1/1.05)[0.75 \cdot 11 + 0.25 \cdot 29] = 14.76190476$. Now the value of the option at time 0 is $V_{0,0} = 1/(1.05)[0.75 \cdot 10.47619048 + 0.25 \cdot 14.76190476] = 10.99773243$ or approximately \$11.

3. (20 points) A gambler plays a coin flipping game in which the probability of winning on a flip is $p = 0.4$ and the probability of losing on a flip is $q = 1 - p = 0.6$. The gambler wants to reach the victory level of \$16 before being ruined with a fortune of \$0. The gambler starts with \$8, bets \$2 on each flip when the fortune is \$6,\$8,\$10 and bets \$4 when the fortune is \$4 or \$12 Compute the probability of ruin in this game.

Solution: Writing the set of first-step equations:

$$\begin{aligned}
 q_{16} &= 0 \\
 q_{12} &= 0.6q_8 + 0.4q_{16} \\
 q_{10} &= 0.6q_8 + 0.4q_{12} \\
 q_8 &= 0.6q_6 + 0.4q_{10} \\
 q_6 &= 0.6q_4 + 0.4q_8 \\
 q_4 &= 0.6q_0 + 0.4q_8 \\
 q_0 &= 1
 \end{aligned}$$

Rewrite the equations as:

$$\begin{aligned}
 0.6q_8 - q_{12} &= 0 \\
 0.6q_8 - q_{10} + 0.4q_{12} &= 0 \\
 0.6q_6 - q_8 + 0.4q_{10} &= 0 \\
 0.6q_4 - q_6 + 0.4q_8 &= 0 \\
 -q_4 + 0.4q_8 &= -0.6
 \end{aligned}$$

In matrix form this is

$$\begin{pmatrix} 0 & 0 & 0.6 & 0 & -1 \\ 0 & 0 & 0.6 & -1 & .4 \\ 0 & 0.6 & -1 & 0.4 & 0 \\ 0.6 & -1 & 0.4 & 0 & 0 \\ -1 & 0 & 0.4 & 0 & 0 \end{pmatrix} \begin{pmatrix} q_4 \\ q_6 \\ q_8 \\ q_{10} \\ q_{12} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ -0.6 \end{pmatrix}.$$

The solution is $q_4 = 0.90857$, $q_6 = 0.85371$, $q_8 = 0.77143$, $q_{10} = 0.64800$, $q_{12} = 0.46286$.

4. (20 points) Suppose that in a certain district, 40% of the registered voters prefer candidate A. A random sample of 50 registered voters is selected. Let S_{50} denote the number in the sample who prefer A. Create a simple probability model for S_{50} . Find the approximate probability that S_{50} is less than 19.

The model is to assign random variables $X_j = 1$ with probability 0.4 if a randomly selected voter prefers candidate A, $X_j = 0$ with probability 0.6 if the voter does not prefer A. Then

$$S_{50} = \sum_{j=1}^{50} X_j$$

is the total of the voters preferring A. Note that $\mathbb{E}[X_j] = 0.40$ and $\text{Var}[X_i] = pq = 0.4 \cdot 0.6 = 0.24$. Then $\mathbb{P}[S_{50} < 19] = \mathbb{P}[(S_{50} - 50 \cdot 0.4)/(\sqrt{50 \cdot 0.24}) < (18.5 - 50 \cdot 0.4)/(\sqrt{50 \cdot 0.24})]$ Then

$$\mathbb{P}[S_{50} < 19] \approx \mathbb{P}[Z < -0.4330127019] = 0.33250277$$