1. (a) (5 points) Why is Geometric Brownian Motion a better model of the stock market than Brownian motion with drift, where the drift parameter is the rate $r$ of market growth?

Solution: Because Geometric Brownian motion is always positive, while there is a positive probability (albeit small) that Brownian Motion with drift can take on negative values.

(b) (10 points) Using the Black-Scholes formula, write the formula for the value of a straddle, a derivative composed of short one share, and long two calls with strike price $K$.

Solution:

\[-S + 2 \left( S \Phi \left( \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) - Ke^{-r(T-t)} \Phi \left( \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma \sqrt{T-t}} \right) \right)\]

(c) (5 points) What mathematical property of the Black-Scholes equation allows you to write the formula for the value of a straddle, as above?

Solution: The Black-Scholes equation is linear, so that the linear combinations of solutions is again a solution.
2. (a) (10 points) Consider a random process defined by the stochastic differential equation with initial condition:

\[ dX = \mu \, dt + \sigma \, dW \]

\[ X(0) = 5 \]

where \( \mu = 2 \) and \( \sigma = 3 \) for \( 0 \leq t \leq 3 \). What is the distribution of the value of the stochastic process at \( t = 3 \)?

**Solution:** \( X(t) \sim N(5 + 2t, 3^2 \cdot t) \) so \( X(3) \sim N(5 + 2 \cdot 3, 3^2 \cdot 3) = N(11, 27) \)

(b) (10 points) Now consider a random process defined by the stochastic differential equation with initial condition:

\[ dX = \mu \, dt + \sigma \, dW \]

\[ X(0) = 5 \]

where \( \mu = 2 \) and \( \sigma = 3 \) for \( 0 \leq t \leq 3 \) and then \( \mu = 3 \) and \( \sigma = 4 \) for \( 3 < t \leq 6 \). What is the distribution of the value of the stochastic process at \( t = 6 \)?

**Solution:** For \( t \geq 3 \), \( X(t) \sim N(5 + 2 \cdot 3, 3^2 \cdot 3) + N(3t, 4^2 \cdot t) = N(11 + 3(t-3), 27 + 4^2 \cdot (t-3)) \). Then \( X(6) \sim N(20, 75) \).

3. Suppose that a stock price \( X(t) \) with current price $50 follows the stochastic differential equation

\[ dX = 0.16X \, dt + 0.30X \, dW \]

\[ X(0) = 50 \]

so the expected return is 16% per year and the volatility is 30% per year. Calculate the following:

(a) (5 points) The expected stock price at the end of the next day. (Use 365 days in a year.)

(b) (5 points) The standard deviation of the stock price at the end of the next day.

(c) (5 points) An interval of stock prices around the mean such that the probability that the stock price lies in this interval has probability at least 0.95 (Hint: Use Chebyshev’s inequality.)
Solution: The Geometric Brownian Motion that is the solution of this SDE is $X(t) = 50 \exp((0.16 - (0.30)^2/2)t + 0.30W(t))$, that is a Geometric Brownian Motion with initial value $z_0 = 50$, and parameters $\mu = (0.16 - (0.30)^2/2)$ and $\sigma = 0.30$. The mean of this Geometric Brownian Motion one day later is 

$$z_0 \exp(\mu t + (1/2)\sigma^2 t) = 50 \cdot \exp(0.16 \cdot (1/365)) = 50.02192260.$$ 

The variance of this Geometric Brownian Motion one day later is 

$$z_0^2 \exp(2\mu t + \sigma^2 t)(\exp(\sigma^2 t) - 1) = 50^2 \exp(2 \cdot 0.16 \cdot (1/365))(\exp(0.30^2 \cdot (1/365)) - 1) = 0.6170557435$$ 

Then the standard deviation is $\sqrt{0.6170557435} = 0.7855289578$. 

Then Chebyshev’s inequality says 

$$\Pr[|X - \mu| \geq k] \leq \sigma^2/k^2$$ 

so 

$$\Pr[|X - \mu| \leq k] = 1 - \Pr[|X - \mu| \geq k] \geq 1 - \sigma^2/k^2.$$ 

Since we want the probability to be 95%, solve $1 - \sigma^2/k^2 = 0.95$ for $k$ to obtain $k = 3.512992296$. Then the confidence interval bounds are: $[50.02192260 - 3.512992296, 50.02192260 + 3.512992296]$, or more properly $[46.50893030, 53.53491490]$. 


4. (20 points) Find a numerical approximation at \( t = 0.2, 0.4, 0.6, 0.8, 1.0 \) to the solution of the Stochastic Differential Equation:

\[
    dX = (1 - X)dt + dW, \quad X(0) = 0.5
\]

*(Remark: With some general parameters, this stochastic differential equation is a model of a “mean-reverting process” called the Ornstein-Uhlenbeck process, a useful model in physics, mathematics, statistics, and finance.)* Use \( dt = 0.2 \), the table of net totals of randomly generated coin flips below, and recall that as in the notes \( dW \approx \sqrt{dt}(S(Ndt)/\sqrt{Ndt}) \).

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<tr>
<th>( j )</th>
<th>( t_j )</th>
<th>( X_j )</th>
<th>( (1 - X_j) )</th>
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<th>( (1 - X_j) \ dt + dW )</th>
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*SOLUTION:* We will use \( dW \approx (S_{100}(100 \cdot 0.2)/\sqrt{100 \cdot 0.2})\sqrt{0.2} \) and take the increments between coinflips 0 and 20, 20 and 40, etc. to 80 and 100.

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5. (5 points) Two call options are identical in all values and parameters except they are written on two different stocks (stock 1 and stock 2) with different volatilities respectively $\sigma_1$ and $\sigma_2$, with $\sigma_1 < \sigma_2$. Which call option will be worth more? Explain why, with full credit given for using the appropriate “Greek”.

*Solution:* The call option on stock 2 will be worth more, since Vega is always positive. As in the notes: “An increase in the volatility will lead to a corresponding increase in the call option value.”
6. (15 points) Assume that a company’s cash position, measured in millions of dollars, follows a general Brownian motion with a drift rate of 0.1 per month, and a volatility rate of 0.16 per month. The initial cash position is 2.0. That is, the cash position at time \( t \) follows the SDE

\[
\begin{align*}
    dX &= 0.1 \, dt + 0.16 \, dW \\
    X(0) &= 2.0
\end{align*}
\]

(a) What are the probability distributions of the cash position after 6 months, and 1 year?
(b) What are the probabilities of a negative cash position at the end of 6 months and one year?

(a) The cash position at 1 month, 6 months, and 1 year are respectively

\[
\begin{align*}
    X(1) &\sim N(2.1, (0.16)^2 \cdot 1) = N(2.1, 0.0256) \\
    X(6) &\sim N(2.6, (0.16)^2 \cdot 6) = N(2.6, 0.1536) \\
    X(12) &\sim N(3.2, (0.16)^2 \cdot 12) = N(3.2, 0.3072)
\end{align*}
\]

(b)

\[
\begin{align*}
    \Pr[X(1) < 0] &= \Pr[(Z - 2.1)/0.16 < -2.1/0.16] = 1.1E - 38 \\
    \Pr[X(6) < 0] &= \Pr[(Z - 2.6)/(0.16\sqrt{6}) < -2.6/(0.16\sqrt{6})] \approx 0 \\
    \Pr[X(12) < 0] &= \Pr[(Z - 3.2)/(0.16\sqrt{12}) < -3.2/(0.16\sqrt{12})] \approx 3.9E - 9
\end{align*}
\]

7. (10 points) Answer True or False, and if false, give a reason why it is false in a single sentence. (If false, 1 point for the answer, 1 point for the reason.)

(a) True or False: The Black-Scholes pricing equation is based on the model that the underlying stock price follows a Brownian Motion.

\textit{False, the underlying stock price is assumed to follow a Geometric Brownian motion, a different (but related) stochastic process.}
(b) True or False: The Black-Scholes pricing equation values an option by taking the present value of the expected return on the option.

*False, the Black Scholes equation prices an option by finding an equivalent dynamic portfolio in the underlying security and a bond and then uses the principle of no-arbitrage to find the equivalent price.*

(c) True or False: The closed form solution of the partial differential equation that we call the Black-Scholes formula represents the final word in financial theory.

*False, it is the starting point and initial ground-breaking idea that stimulated the whole area of mathematical finance.*

(d) True or False: The volatility of a stock price can be estimated from the Black-Scholes Formula if the option values are known from the market.

*True, that is called the implied volatility.*

(e) True or False: European puts cannot be valued from the Black-Scholes equation, only European calls.

*False, they can by solving a different terminal value problem, one that we did not choose to solve, but could have.*