1. (a) (5 points) Why is Geometric Brownian Motion a better model of the stock market than Brownian motion with drift, where the drift parameter is the rate $r$ of market growth?

(b) (10 points) Using the Black-Scholes formula, write the formula for the value of a straddle, a derivative composed of short one share, and long two calls with strike price $K$.

(c) (5 points) What mathematical property of the Black-Scholes equation allows you to write the formula for the value of a straddle, as above?
2. (a) (10 points) Consider a random process defined by the stochastic differential equation with initial condition:

\[ dX = \mu \, dt + \sigma \, dW \]
\[ X(0) = 5 \]

where \( \mu = 2 \) and \( \sigma = 3 \) for \( 0 \leq t \leq 3 \). What is the distribution of the value of the stochastic process at \( t = 3 \)?

(b) (10 points) Now consider a random process defined by the stochastic differential equation with initial condition:

\[ dX = \mu \, dt + \sigma \, dW \]
\[ X(0) = 5 \]

where \( \mu = 2 \) and \( \sigma = 3 \) for \( 0 \leq t \leq 3 \) and then \( \mu = 3 \) and \( \sigma = 4 \) for \( 3 < t \leq 6 \). What is the distribution of the value of the stochastic process at \( t = 6 \)?

3. Suppose that a stock price \( X(t) \) with current price $50 follows the stochastic differential equation

\[ dX = 0.16X \, dt + 0.30X \, dW \]
\[ X(0) = 50 \]

so the expected return is 16% per year and the volatility is 30% per year. Calculate the following:

(a) (5 points) The expected stock price at the end of the next day. (Use 365 days in a year.)

(b) (5 points) The standard deviation of the stock price at the end of the next day.

(c) (5 points) An interval of stock prices around the mean such that the probability that the stock price lies in this interval has probability at least 0.95 (Hint: Use Chebyshev’s inequality.)
4. (20 points) Find a numerical approximation at \( t = 0.2, 0.4, 0.6, 0.8, 1.0 \) to the solution of the Stochastic Differential Equation:

\[
\frac{dX}{dt} = (1 - X)dt + dW, \quad X(0) = 0.5
\]

(Remark: With some general parameters, this stochastic differential equation is a model of a “mean-reverting process” called the Ornstein-Uhlenbeck process, a useful model in physics, mathematics, statistics, and finance.) Use \( dt = 0.2 \), the table of net totals of randomly generated coin flips below, and recall that as in the notes \( dW \approx \sqrt{dt} \left( \frac{S(Ndt)}{\sqrt{Ndt}} \right) \).

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<th>( j )</th>
<th>( t_j )</th>
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<th>( dW )</th>
<th>( (1 - X_j) ) ( dt + dW )</th>
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\[
\begin{array}{ccccccccc}
 n & 0 & 10 & 20 & 30 & 40 & 50 & 60 & 70 & 80 & 90 & 100 \\
 S_n & 0 & 0 & 2 & -4 & -6 & -8 & -10 & -10 & -12 & -12 \\
\end{array}
\]

5. (5 points) Two call options are identical in all values and parameters except they are written on two different stocks (stock 1 and stock 2) with different volatilities respectively \( \sigma_1 \) and \( \sigma_2 \), with \( \sigma_1 < \sigma_2 \). Which call option will be worth more? Explain why, with full credit given for using the appropriate “Greek”.


6. (15 points) Assume that a company’s cash position, measured in millions of dollars, follows a general Brownian motion with a drift rate of 0.1 per month, and a variance rate of 0.16 per month. The initial cash position is 2.0. That is, the cash position at time $t$ follows the SDE

$$dX = 0.1 \, dt + 0.16 \, dW$$

$$X(0) = 2.0$$

(a) What are the probability distributions of the cash position after 6 months, and 1 year?

(b) What are the probabilities of a negative cash position at the end of 6 months and one year?

7. (10 points) Answer True or False, and if false, give a reason why it is false in a single sentence. (If false, 1 point for the answer, 1 point for the reason.)

(a) True or False: The Black-Scholes pricing equation is based on the model that the underlying stock price follows a Brownian Motion.

(b) True or False: The Black-Scholes pricing equation values an option by taking the present value of the expected return on the option.

(c) True or False: The closed form solution of the partial differential equation that we call the Black-Scholes formula represents the final word in financial theory.

(d) True or False: The volatility of a stock price can be estimated from the Black-Scholes Formula if the option values are known from the market.

(e) True or False: European puts cannot be valued from the Black-Scholes equation, only European calls.