Math 496/889 **Exam 1** **Name:**________________
Friday, November 7, 2003 **SSN:**________________

Problem	1	2	3	4	5	6	Total
Possible	20	10	15	10	15	20	90
Points							

1. (20 points) Consider a two-time-stage binomial-outcome model for pricing an financial derivative. Each time stage is a year. A stock starts at 100. In each year, the stock can go up by 8% or down by 4%. The continously compounded interest rate on a \$1 bond is constant at 4% each year. Find the price of a *put option* with exercise price 105, with exercise date at the end of the second year. Also, find the replicating portfolio at each node.

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                    At node (1,1)            At node (2,2)
                    phi = -0.11085           V = 0
                    psi = 11.414             S = 116.64
                    V = 0.414
                    S = 108
At node (0,0)                                At node (2,1)
phi = -0.372                                 V = 1.32
psi = 39.042                                 S = 103.68
V = 1.80
S = 100
                   At node (1,0)            At node (2,0)
                   phi = -1                 V = 12.84
                   \psi = 100.883           S = 92.16
                   V = 4.883
                   S = 96


  Pi = 0.67342,  1 - pi = 0.32657
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2. (10 points) James Bond is determined to ruin the casino at Monte Carlo by betting \$1 on Red at the roulette wheel. The probability of Bond winning at one turn in this game is $18/38 \approx 0.474$. James Bond, being Agent 007, is backed by the full financial might of the British Empire, and so can be considered to have unlimited funds. Approximately how much money should the casino have to start with so that Bond has only a "one-in-a-million" chance of ruining the casino?

   Solution: Here the casino is the "gambler", with probability of success at any trial being $p = 0.526$ and the probability of failure at any trial is $q = 1 - p = 0.474$. The probability of ultimate ruin of a gambler playing against an infinitely rich adversary is is (if $p > q$), $(q/p)_0^S$ and we wish to make this $10^(- 6)$. Solve $(0.474/0.526)_0^S = 10^(- 6)$ for $S_0 = 132.72$. So if the casino has \$133 it has less than a 1-in-a-million chance of being ruined.

3. (15 points) Make a convincing argument that the net fortune S_n coin-flipping game must be 0 infinitely many times. (That is, we consider the sum S_n where the independent, identically

distributed random variables in the sum $S_n = X_1 + ... + X_n$ are the Bernoulli random variables $X_i = +1$ with probability $p = 1/2$ and $X_i = -1$ with probability $q = 1 - p = 1/2$.)
Suggestion: Consider using the results of the Law of the Iterated Logarithm.

Solution: Take $\lambda_{-1} = 1 - 1/2 < 1$. The Law of the iterated logarithm says that that there is a sequence times going to infinity when the fortune is greater than $(1-1/2) * \sqrt{2t\log(\log(t))}$ hence positive, and a sequence of times going to infinity when the fortune is less than $-(1-1/2) * \sqrt{2t\log(\log(t))}$ hence negative. By inter-leaving the selection of times (possible because both sequences of times are going to infinity) we can find a sequence of ineredíate times when the value must be zero. This problem can also be done more rigorously and simply with the Borel-Cantelli lemma.

4. (10 points) For two random variables X and Y, statisticians call

$$\mathrm{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$

the *covariance* of X and Y. If X and Y are independent, then $\mathrm{Cov}(X, Y) = 0$. A positive value of $\mathrm{Cov}(X, Y)$ indicates that Y tends to increase as X increases, while a negative value indicates that Y tends to decrease when X increases. Thus, $\mathrm{Cov}(X, Y)$ is an indication of the mutual dependence of X and Y. Show that

$$\mathrm{Cov}(W(s), W(t)) = E[W(s)W(t)] = s$$

for $0 < s < t$.

Solution: Write

$$\begin{aligned}
\mathrm{Cov}(W(s), W(t)) &= E[W(s)W(t)] \\
&= E[W(s) \cdot ((W(t) - W(s)) + (W(s))] \\
&= E[W(s) \cdot ((W(t) - W(s))] + E[W(s) \cdot W(s))] \\
&= 0 + s = s.
\end{aligned}$$

5. (15 points) Let Z be a normally distributed random variable, with mean 0 and variance 1, that is, $Z \sim N(0, 1)$. Then consider the continuous time stochastic process $X(t) = \sqrt{t}Z$. Show that the distribution of $X(t)$ is normal with mean 0 with variance t. Is $X(t)$ a Brownian motion? Explain why or why not.

Solution: No because $X(s) = \sqrt{s}Z$ is not independent of $X(t) - X(s) = (\sqrt{t} - \sqrt{s})Z$ Furthermore, the increments have variance $(\sqrt{t} - \sqrt{s})^2 = t - 2\sqrt{ts} + s \neq t - s$.

6. (20 points) Simulate the solution of the stochastic differential equation

$$dY(t) = Y(t)dt + Y(t)dW, \qquad Y(0) = 1$$

on the interval $[0, 1]$ with a step size $dt = 1/5$. Use $W_{25}(t)$ with increments of $1/5$ to approximate Brownian Motion.