Monte Carlo Methods for Integration

A Project for Math 489/889

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Suppose we wish to numerically evaluate $\int_0^{\frac{\pi}{2}} \cos(x)\, \mathrm{d}x$ reliably to some specified error. We will evaluate this integral with basic calculus. We will also pretend that this is an integral which is too complicated to solve analytically and evaluate it with the Monte Carlo method. Then we can compare the results to understand the error of the Monte Carlo aproach. We notice
that we could consider

$$\int_0^{\frac{\pi}{2}} \cos(x)\, \mathrm{d}x = \frac{\pi}{2} \int_0^{\frac{\pi}{2}} \cos(x) \cdot \frac{2}{\pi}\, \mathrm{d}x$$

and that

$$\mathrm{E}[\cos(x)] = \int_0^{\frac{\pi}{2}} \cos(x) \cdot \frac{2}{\pi}\, \mathrm{d}x$$

is the mean of $\cos(X)$ where X is a uniformly distributed random variable on $[0, \pi/2]$.
Therefore

$$J = \frac{\pi}{2} \mathrm{E}[\cos(X)].$$

But we have another way to evaluate the mean and therefore the integral J, namely

$$J_n = \frac{\pi}{2} \sum_{k=1}^{n} \cos(X_k) \quad X_k\ [0, \pi/2].$$

Analytic Investigation of the Integral, using Calculus

$$\mu := \int_0^{\frac{\pi}{2}} \cos(x) \left(\frac{2}{\pi}\right) dx$$

$$\frac{2}{\pi} \qquad \textbf{(1.1)}$$

$\xrightarrow{\text{at 5 digits}}$

$$0.63662 \qquad \textbf{(1.2)}$$

$$sigmaSquared := \int_0^{\frac{\pi}{2}} (\cos(x) - \mu)^2 \left(\frac{2}{\pi}\right) dx$$

$$\frac{1}{2}\,\frac{\pi^2 - 8}{\pi^2} \qquad \textbf{(1.3)}$$

$\xrightarrow{\text{at 5 digits}}$

$$0.094720 \qquad \textbf{(1.4)}$$

Symbolic expression of the mean and the variance of J_n

The symbolic expression of J_n in terms of μ is:

$$E[J_n] = E\left[\frac{\pi}{2} \cdot \frac{1}{n} \cdot \sum_{k=1}^{n} \cos(X_k)\right] = \frac{\pi}{2} \cdot \frac{1}{n} \cdot n \cdot E[\cos(X_k)] = \frac{\pi}{2} \cdot \mu = J$$

Likewise the symbolic expression of the variance is:

$$Var[J_n] = Var\left[\frac{\pi}{2} \cdot \frac{1}{n} \cdot \sum_{k=1}^{n} \cos(X_k)\right] = \frac{\pi^2}{4} \cdot \frac{1}{n^2}\, n \cdot Var[\cos(X_k)] = \frac{\pi^2}{4} \cdot \frac{\sigma^2}{n}$$

Justification of normality

Because J_n is the sum of n independent random variables, then by the Central Limit Theorem, we expect that for large n

$$\frac{(J_n - E[J_n])}{\sqrt{Var(J_n)}}$$

is normally distributed with mean 0 and variance 1.
Then using the results derived above, we expect that

$$\frac{(J_n - J)}{\sqrt{Var(J_n)}}$$

is approximately normally distibuted with mean 0 and variance 1.

Finding the confidence interval values

Therefore asking that

$$Pr\left[\left|\frac{(J_n - J)}{\sqrt{Var(J_n)}}\right| < z\right] > 0.99$$

or

$$Pr\left[|Z| < z\right\} > 0.99$$

is the same as requiring z > 2.57582. We can evaluate this with a little examination of the symmetric normal probability distribution function and Maple numerics:

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with(Statistics) :
Quantile(Normal(0, 1), 0.995)
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2.575829304 **(4.1)**

Using the confidence interval to estimate the required number of trials

We desire $|J_n - J| < 0.05$ with probability greater than 0.99. This would be equivalent to finding *n* so large that

$$\left|\frac{(J_n - J)}{\sqrt{Var(J_n)}}\right| < \frac{0.05}{\sqrt{\frac{\pi^2}{4} \cdot \frac{\sigma^2}{n}}} = \frac{0.05}{\sqrt{\frac{\pi^2}{4} \cdot \frac{\frac{1}{2}\,\frac{\pi^2 - 8}{\pi^2}}{n}}}$$

$$\frac{0.05}{\sqrt{\frac{\frac{\pi^2}{4}\cdot\frac{1}{2}\frac{\pi^2-8}{\pi^2}}{n}}} > 2.57582 \xrightarrow{\text{solve for n}} [[620.2271139 < n]]$$

So if we choose more than 621 samples, we should obtain an approximation that has an error of less than 0.05 with probability greatly than 0.99.

Experiment to determine J

$X := Uniform\left(0, \frac{\pi}{2}\right)$

$$Uniform\left(0, \frac{1}{2}\pi\right) \quad \textbf{(6.1)}$$

$samp := Sample(X, 621);$

$$\left[\begin{array}{c} \textit{1 .. 621 Vector}_{row} \\ \textit{Data Type: float}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right] \quad \textbf{(6.2)}$$

$cosSample := map(\cos, samp) :$
$montecarloApproximation := Mean(cosSample)$

$$0.6322092758 \quad \textbf{(6.3)}$$

$absoluteError := evalf\left(\text{abs}\left(\frac{\pi}{2}\cdot\mu - \frac{\pi}{2}\cdot montecarloApproximation\right)\right)$

$$0.0069279917 \quad \textbf{(6.4)}$$

Our deliberations above tell us we could repeat this experiment 100 times and expect to get a similar result at least 99 times.

Experiment to determine the integral of sine

We believe that $\int_0^{\frac{\pi}{2}} \sin(x)\,dx$ should be very similar to $\int_0^{\frac{\pi}{2}} \cos(x)\,dx$ and the distribution of $\sin(X)$ should be similar to $\cos(X)$ for X uniformly distributed on $\left[0, \frac{\pi}{2}\right]$.

That means our inution tells us that we should use a sample size of 621 or greater.
$samp := Sample(X, 621);$

$$\left[\begin{array}{c} \textit{1 .. 621 Vector}_{row} \\ \textit{Data Type: float}_8 \\ \textit{Storage: rectangular} \\ \textit{Order: Fortran\_order} \end{array}\right] \quad (7.1)$$

$sinSample := map(\sin, samp) :$

$montecarloApproximationSine := \frac{\pi}{2} \cdot Mean(sinSample)$

$$0.3091753322\,\pi \quad (7.2)$$

$\xrightarrow{\text{at 5 digits}}$

$$0.97132 \quad (7.3)$$

$\int_0^{\frac{\pi}{2}} \sin(x)\,dx$

$$1 \quad (7.4)$$