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Math 489/Math 889
Stochastic Processes and
Advanced Mathematical Finance
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Stochastic Differential Equations and the Euler-Maruyama Method

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Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.

Section Starter Question

Explain how to use a slope-field diagram to solve the ordinary differential equation

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x.$$

How would you turn that process into an algorithm to numerically compute an approximate solution without a diagram?

Key Concepts

1. We can numerically simulate the solution to stochastic differential equations with an analog to Euler's method, called the Euler-Maruyama (EM) method.

Vocabulary

1. A **stochastic differential equation** is a mathematical equation relating a stochastic process to its local deterministic and random components. The goal is to extend the relation to find the stochastic process. Under mild conditions on the relationship, and with a specifying initial condition, solutions of stochastic differential equations exist and are unique.

2. The **Euler-Maruyama (EM) method** is a numerical method for simulating the solutions of a stochastic differential equation based on

the definition of the Itô stochastic integral: Given

$$\mathrm{d}X(t) = G(X(t))\,\mathrm{d}t + H(X(t))\,\mathrm{d}W(t), \qquad X(t_0) = X_0,$$

and a step size $\mathrm{d}t$, we approximate and simulate with

$$X_j = X_{j-1} + G(X_{j-1})\,\mathrm{d}t + H(X_{j-1})(W(t_{j-1} + \mathrm{d}t) - W(t_{j-1}))$$

3. Extensions and variants of Standard Brownian Motion defined through stochastic differential equations are **Brownian Motion with drift**, **scaled Brownian Motion**, and **geometric Brownian Motion**.

Mathematical Ideas

Stochastic Differential Equations: Symbolically

The straight line segment is the building block of differential calculus. The basic idea behind differential calculus is that differentiable functions, no matter how difficult their global behavior, are locally approximated by straight line segments. In particular, this is the idea behind Euler's method for approximating differentiable functions defined by differential equations.

We know that rescaling ("zooming in" on) Brownian motion does not produce a straight line, it produces another image of Brownian motion. This self-similarity is ideal for an infinitesimal building block, for instance, we could build global Brownian motion out of lots of local "chunks" of Brownian motion. This suggests we could build other stochastic processes out of suitably scaled Brownian motion. In addition, if we include straight line segments we can overlay the behavior of differentiable functions onto the stochastic processes as well. Thus, straight line segments and "chunks" of Brownian motion are the building blocks of stochastic calculus.

With stochastic differential calculus, we can build a nice class of new stochastic processes. We do this by specifying how to build the new stochastic processes locally from our base deterministic function, the straight line and our base stochastic process, Standard Brownian Motion. We write the local change in value of the stochastic process over a time interval of (infinitesimal) length $\mathrm{d}t$ as

$$\mathrm{d}X = G(X(t))\ \mathrm{d}t + H(X(t))\ \mathrm{d}W(t), X(t_0) = X_0. \qquad (1)$$

Note that we are not allowed to write

$$\frac{\mathrm{d}X}{\mathrm{d}t} = G(X(t)) + H(X(t))\frac{\mathrm{d}W}{\mathrm{d}t}, X(t_0) = X_0$$

since Standard Brownian Motion is nowhere differentiable with probability 1. (Actually, the informal stochastic differential equation is a compact way of writing a rigorously defined, equivalent implicit Itô integral equation. Since we do not have the required rigor, we will approach the stochastic differential equation intuitively.)

The stochastic differential equation says the initial point (t_0, X_0) is specified, perhaps with X_0 a random variable with a given distribution. A deterministic component at each point has a slope determined through G at that point. In addition, there is some random perturbation that effects the evolution of the process. The random perturbation is normally distributed with mean 0. The variance of the random perturbation is $(H(X(t)))^2$ at $(t, X(t))$. This is a simple expression of a Stochastic Differential Equation (SDE) which determines a stochastic process, just as an Ordinary Differential Equation (ODE) determines a differentiable function. We extend the process with the incremental change information and repeat. This is an expression in words of the **Euler-Maruyama method** for numerically simulating the stochastic differential expression.

Example. The simplest stochastic differential equation is

$$\mathrm{d}X = r\ \mathrm{d}t + \mathrm{d}W, \qquad X(0) = b$$

where r is a constant. Take a deterministic initial condition to be $X(0) = b$. This process is the stochastic extension of the differential equation expression of a straight line. The new stochastic process X is drifting or trending at rate r with a random variation due to Brownian Motion perturbations around that trend. We will later show explicitly that the solution of this SDE is $X(t) = b + rt + W(t)$ although it is seems intuitively clear that this should be the process. We will call this **Brownian motion with drift**.

Example. The next simplest stochastic differential equation is

$$dX = \sigma \, dW, \qquad X(0) = b$$

This stochastic differential equation says that the process is evolving as a multiple of Standard Brownian Motion. The solution may be easily guessed as $X(t) = \sigma W(t)$ which has variance $\sigma^2 t$ on increments of length t. Sometimes this is called Brownian Motion (in contrast to Standard Brownian Motion which has variance t on increments of length t).

We combine the previous two examples to consider

$$dX = r \, dt + \sigma \, dW, \qquad X(0) = b$$

which has solution $X(t) = b + rt + \sigma W(t)$, a **multiple of Brownian Motion with drift r started at b**. Sometimes this extension of Standard Brownian motion is called Brownian Motion. Some authors consider this process directly instead of the more special case we considered in the previous chapter.

Example. The next simplest and first non-trivial differential equation is $dX = X \, dW$. Here the differential equation says that process is evolving like Brownian motion with a variance which is the square of the process value. When the process is small, the variance is small, when the process is large, the variance is large. Expressing the stochastic differential equation as $dX/X = dW$ we may say that the relative change acts like Standard Brownian Motion. The resulting stochastic process is called **geometric Brownian motion** and it will figure extensively in what we consider later as models of security prices.

Example. The next simplest differential equation is

$$dX = rX \, dt + \sigma X \, dW, \qquad X(0) = b.$$

Here the stochastic differential equation says that the growth of the process at a point is proportional to the process value, with a random perturbation proportional to the process value. Again looking ahead, we could write the differential equation as $dX/X = r \, dt + \sigma \, dW$ and interpret it to say the relative rate of increase is proportional to the time observed together with a random perturbation like a Brownian increment corresponding to the length of time. We will show later that the analytic expression for the stochastic process defined by this SDE is $b \exp((r - \frac{1}{2}\sigma^2)t + \sigma W(t))$.

Stochastic Differential Equations: Numerically

The sample path that the Euler-Maruyama method produces numerically is the analog of using the Euler method.

The formula for the Euler-Maruyama (EM) method is based on the definition of the Itô stochastic integral:

$$X_j = X_{j-1} + G(X_{j-1})\,\mathrm{d}t + H(X_{j-1})(W(t_{j-1}+\mathrm{d}t) - W(t_{j-1})), \qquad t_j = t_{j-1} + \mathrm{d}t\,.$$

Note that the initial conditions X_0 and t_0 set the starting point.

In this text, we do not use Brownian motion directly to obtain the increments $W(t_{j-1}+\mathrm{d}t) - W(t_{j-1})$ since we don't have a direct source of values of Brownian Motion. Instead we use coin-flipping sequences of an appropriate length to create an approximation to $W(t)$. Note that since the increments $W(t_{j-1}+\mathrm{d}t) - W(t_{j-1})$ are independent and identically distributed, we will use independent coin-flip sequences to generate the approximation of the increments. That is,

$$dW = W(t_{j-1}+\mathrm{d}t) - W(t_{j-1}) = W(\mathrm{d}t) \approx \hat{W}_N(\mathrm{d}t) = \frac{\hat{W}(N\ \mathrm{d}t)}{\sqrt{N}} = \sqrt{\mathrm{d}t}\frac{\hat{W}(N\ \mathrm{d}t)}{\sqrt{N\ \mathrm{d}t}}.$$

The first equality above is the definition of an increment, the second equality means the random variables $W(t_{j-1}+\mathrm{d}t) - W(t_{j-1})$ and $W(\mathrm{d}t)$ have the same distribution because of the definition of Standard Brownian Motion which specifies that increments with equal length are normally distributed with variance equal to the increment length. The approximate equality occurs because of the approximation of Brownian Motion by coin-flipping sequences. We generate the approximations using a random number generator, but we could as well use actual coin-flipping. In the table below the generation of the sequences is not recorded, only the summed and scaled (independently sampled) outcomes. For convenience, take $\mathrm{d}t = 1/10$, $N = 100$, so we need $\hat{W}(100 \cdot (1/10))/\sqrt{100} = T_{10}/10$. Then to obtain the entries in the column labeled dW in the table we flip a coin 10 and record $T_{10}/10$. Take $r = 2$, $b = 1$, and $\sigma = 1$, so we simulate the solution of

$$\mathrm{d}X = 2X\ \mathrm{d}t + X\ \mathrm{d}W, \qquad X(0) = 1.$$

j	t_j	X_j	$2X_j\ \mathrm{d}t$	$\mathrm{d}W$	$X_j\,\mathrm{d}W$	$2X_j\,\mathrm{d}t + X_j\,\mathrm{d}W$	$X_j + 2X_j\ \mathrm{d}t + X_j\,\mathrm{d}W$
0	0	1	0.2	0	0	0.2	1.2
1	0.1	1.2	0.24	0.2	0.24	0.48	1.68
2	0.2	1.68	0.34	-0.2	-0.34	0.0	1.68
3	0.3	1.68	0.34	0.4	0.67	1.01	2.69
4	0.4	2.69	0.54	-0.2	-0.54	0.0	2.69
5	0.5	2.69	0.54	0	0	0.54	3.23
6	0.6	3.23	0.65	0.4	1.29	1.94	5.16
7	0.7	5.16	1.03	0.4	2.06	3.1	8.26
8	0.8	8.26	1.65	0.4	3.3	4.95	13.21
9	0.9	13.21	2.64	0	0	2.64	15.85
10	1.0	15.85					

Of course, this can be programmed and the step size made much smaller, presumably with better approximation properties. In fact, it is possible to consider kinds of convergence for the EM method comparable to the Strong Law of Large Numbers and the Weak Law of Large Numbers. See the Problems for examples.

Discussion

The numerical approximation procedure using coin-flipping makes it clear that the Euler-Maruyama method generates a random process. The value of the process depends on the time value and the coin-flip sequence. Each generation of an approximation will be different because the coin-flip sequence is different. The Euler-Maruyama method generates a stochastic process path approximation. To derive distributions and statistics about the process requires generating multiple paths, see the Problems for examples.

This shows that stochastic differential equations provide a way to define new stochastic processes. This is analogous to the notion that ordinary differential equations define new functions which can then be studied and used. In fact, one approach to developing calculus and the analysis of functions is to start with differential equations, use the Euler method to define approximations of solutions, and then to develop a theory to handle the passage to continuous variables. This approach is especially useful for a mathematical modeling viewpoint since the modeling often is expressed in differential equations.

This approach of starting with stochastic differential equations to describe a situation and numerically defining new stochastic processes to model the situation is followed in this text. At certain points, we appeal to more rigorous mathematical theory to justify the modeling and approximation. One important justification is to assert that if we write a stochastic differential equation, then solutions exist and the stochastic differential equation always yields the same process under equivalent conditions. The Existence-Uniqueness Theorem shows that under reasonable modeling conditions stochastic differential equations do indeed satisfy this requirement.

Theorem 1 (Existence-Uniqueness)**.** *For the stochastic differential equation*

$$\mathrm{d}X = G(t, X(t))\,\mathrm{d}t + H(t, X(t))\,\mathrm{d}W(t), \qquad X(t_0) = X_0$$

assume

1. *Both $G(t,x)$ and $H(t,x)$ are continuous on $(t,x) \in [t_0, T] \times \mathbb{R}$.*
2. *The coefficient functions G and H satisfy a Lipschitz condition:*

   $$|G(t,x) - G(t,y)| + |H(t,x) - H(t,y)| \leq K|x - y|.$$

3. *The coefficient functions G and H satisfy a growth condition in the second variable*

   $$|G(t,x)|^2 + |H(t,x)|^2 \leq K(1 + |x|^2)$$

   for all $t \in [t_0, T]$ and $x \in \mathbb{R}$.

Then the stochastic differential equation has a strong solution on $[t_0, T]$ which is continuous with probability 1 and

$$\sup_{t \in [t_0, T]} \mathbb{E}\left[X^2(t)\right] < \infty$$

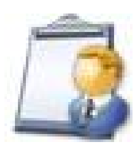
and for each given Wiener process $W(t)$, the corresponding strong solutions are pathwise unique which means that if X and Y are two strong solutions, then

$$\mathbb{P}\left[\sup_{t \in [t_0, T]} |X(t) - Y(t)| = 0\right] = 1.$$

See the reference [4] for a precise definition of “strong solution” but essentially it means that each given Wiener process $W(t)$ we can generate a solution to the SDE. Note that the coefficient functions are here two-variable functions of both time t and location x, which is more general than the functions considered in equation (1). The restrictions on the functions $G(t,x)$ and $H(t,x)$, especially the continuity condition, can be considerably relaxed and the theorem will still remain true.

Sources

This section is adapted from: "An Algorithmic Introduction to the Numerical Simulation of Stochastic Differential Equations", by Desmond J. Higham, in SIAM Review, Vol. 43, No. 3, pp. 525-546, 2001 and *Financial Calculus: An introduction to derivative pricing* by M. Baxter, and A. Rennie, Cambridge University Press, 1996, pages 52-62. The Existence-Uniqueness Theorem is adapted from *An Introduction to Stochastic Processes with Applications to Biology*, by L. J. S. Allen, Pearson Prentice-Hall, 2003, pages 342-343 and *Numerical Solution of Stochastic Differential Equations*, by Peter Kloeden and Eckhard Platen, Springer Verlag, 1992, pages 127-131.



Problems to Work for Understanding

1. Graph an approximation of a multiple of Brownian motion with drift with parameters $b = 2$, $r = 1/2$ and $\sigma = 2$ in the following two ways:

   - Flip a coin 25 times, recording whether it comes up Heads or Tails each time, Scoring $Y_i = +1$ for each Heads and $Y_i = -1$ for each flip, also keep track of the accumulated sum $T_n = \sum_{i=1}^{n} T_i$ for $i = 1 \dots 25$. Using $N = 5$ compute the rescaled approximation $\hat{W}_5(t) = (1/\sqrt{5})T_{5t}$ at the values $t = 0, 1/5, 2/5, 3/5, \dots 24/5, 5$ on $[0, 5]$. Finally compute and graph the value of $X(t) = b + rt + \sigma\hat{W}_5(t)$.
   - Using the same values of $\hat{W}_5(t)$ as approximations for $W(dt)$ compute the values of the solution of the stochastic differential equation $\mathrm{d}X = r\,\mathrm{d}t + \sigma\,\mathrm{d}W, \quad X(0) = b$.

2. Repeat the previous problem with parameters $b = 2$, $r = -1/2$ and $\sigma = 2$.

3. Repeat the previous problem with parameters $b = 2$, $r = 1/2$ and $\sigma = -2$.

4. Simulate the solution of the stochastic differential equation

$$\mathrm{d}X(t) = X(t)\,\mathrm{d}t + 2X(t)\,\mathrm{d}X$$

on the interval $[0, 1]$ with initial condition $X(0) = 1$ and step size $\Delta t = 1/10$.

5. Simulate the solution of the stochastic differential equation

$$\mathrm{d}X(t) = tX(t)\,\mathrm{d}t + 2X(t)\,\mathrm{d}X$$

on the interval $[0, 1]$ with initial condition $X(0) = 1$ and step size $\Delta t = 1/10$. Note the difference with the previous problem, now the multiplier of the $\mathrm{d}t$ term is a function of time.

6. Write a program with parameters r, σ, b, T and N (so $\mathrm{d}t = T/N$) that computes and graphs the approximation of the solution of the stochastic differential equation

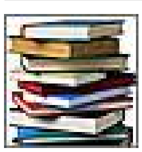
$$\mathrm{d}X(t) = rX(t)\,\mathrm{d}t + \sigma X(t)\,\mathrm{d}X$$

with $X(0) = b$ on the interval $[0, T]$. Apply the program to the stochastic differential equation with $r = 2$, $\sigma = 1$, $b = 1$, and $N = 2^6, 2^7, 2^8$ on the interval $[0, 1]$.

7. Generalize the program from the previous problem to include a parameter M for the number of sample paths computed. Then using this program on the interval $[0, 1]$ with $M = 1000$, and $N = 2^8$ compute $\mathbb{E}\left[|X_n - X(1)|\right]$, where $X(1) = b\mathrm{e}^{r - \frac{1}{2}\sigma^2 + \sigma \hat{W}_{2^8}(1)}$.

8. Using the program from the previous problem with $M = 1000$ and $N = 2^5, 2^6, 2^7, 2^8, 2^9$ compute $\mathbb{E}\left[|X_N - X(1)|\right]$, where $X(1) = b\mathrm{e}^{r - \frac{1}{2}\sigma^2 + \sigma \hat{W}_{2^9}(1)}$. Then for the 5 values of N, make a log-log plot of $\mathbb{E}\left[|X_N - X(1)|\right]$ on the vertical axis against $\Delta t = 1/N$ on the horizontal axis. Using the slope of the resulting best-fit line experimentally determine the order of convergence γ so that

$$\mathbb{E}\left[|X_N - X(1)|\right] \leq C(\Delta t)^\gamma.$$



Reading Suggestion:

References

[1] Linda J. S. Allen. *An Introduction to Stochastic Processes with Applications to biology*. Pearson Prentice-Hall, 2003.

[2] M. Baxter and A. Rennie. *Financial Calculus: An introduction to derivative pricing*. Cambridge University Press, 1996. HG 6024 A2W554.

[3] Desmond J. Higham. An algorithmic introduction to numerical simulation of stochastic differential equations. *SIAM Review*, 43(3):525–546, 2001.

[4] P. Kloeden and E. Platen. *Numerical Solution of Stochastic Differential Equations*, volume 23 of *Stochastic Modelling and Applied Probability*. Springer, 1992.

Outside Readings and Links:

1. Maple Stochastic Package The MAPLE stochastic package offers a number of MAPLE routines for stochastic differential equations.

2. Matlab program files for Stochastic Differential Equations offers a number of MATLAB routines for stochastic differential equations.

I have checked the links to external sites for usefulness. Links to external websites are provided as a convenience. I do not endorse, control, monitor, or guarantee the information contained in any external website. I don't guarantee that the links are active at all times. Use the links here with the same caution as you would all information on the Internet. This website reflects the thoughts, interests and opinions of its author. They do not explicitly represent official positions or policies of my employer.

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