Stochastic Processes and Advanced Mathematical Finance

Itô’s Formula

Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.

Section Starter Question

State the Taylor expansion of a function $f(x)$ up to order 1. What is the relation of this expansion to the Mean Value Theorem of calculus? What is the relation of this expansion to the Fundamental Theorem of calculus?
Key Concepts

1. Itô’s formula is an expansion expressing a stochastic process in terms of the deterministic differential and the Wiener process differential, that is, the stochastic differential equation for the process.

2. Solving stochastic differential equations follows by guessing solutions based on comparison with the form of Itô’s formula.

Vocabulary

1. Itô’s formula is often also called Itô’s lemma by other authors and texts. Some authors believe that this result is more important than a mere lemma, and so I adopt the alternative name of “formula”. “Formula” also emphasizes the analogy with the chain “rule” and the Taylor “expansion”.
Mathematical Ideas

Motivation, Examples and Counterexamples

We need some operational rules that allow us to manipulate stochastic processes with stochastic calculus.

The important thing to know about traditional differential calculus is that it is:

- the Fundamental Theorem of Calculus;
- the chain rule; and
- Taylor polynomials and Taylor series

that enable us to calculate with functions. A deeper understanding of calculus recognizes that these three calculus theorems are all aspects of the same fundamental idea. Likewise we need similar rules and formulas for stochastic processes. Itô’s formula will perform that function for us. However, Itô’s formula acts in the capacity of all three of the calculus theorems, and we have only one such theorem for stochastic calculus.

The next example will show us that we will need some new rules for stochastic calculus, the old rules from calculus will no longer make sense.

Example. Consider the process which is the square of the Wiener process:

\[ Y(t) = W(t)^2. \]

We notice that this process is always non-negative, \( Y(0) = 0 \), \( Y \) has infinitely many zeroes on \( t > 0 \) and \( \mathbb{E} [Y(t)] = \mathbb{E} [W(t)^2] = t \). What more can we say about this process? For example, what is the stochastic differential of \( Y(t) \) and what would that tell us about \( Y(t) \)?

Using naive calculus, we might conjecture using the ordinary chain rule

\[ dY = 2W(t) \, dW(t). \]

If that were true then the Fundamental Theorem of Calculus would imply

\[ Y(t) = \int_0^t dY = \int_0^t 2W(t) \, dW(t). \]
should also be true. But consider \( \int_0^t 2W(t) \, dW(t) \). It ought to correspond to a limit of a summation (for instance a Riemann-Stieltjes left sum):

\[
\int_0^t 2W(t) \, dW(t) \approx \sum_{i=1}^{n} 2W((i-1)t/n)[W(it/n) - W((i-1)t/n)]
\]

But look at this carefully: \( W((i-1)t/n) = W((i-1)t/n) - W(0) \) is independent of \( [W(it/n) - W((i-1)t/n)] \) by property 2 of the definition of the Wiener process. Therefore, if what we conjecture is true, the expected value of the summation will be zero:

\[
\mathbb{E}[Y(t)] = \mathbb{E} \left[ \int_0^t 2W(t) \, dW(t) \right] \\
= \mathbb{E} \left[ \lim_{n \to \infty} \sum_{i=1}^{n} 2W((i-1)t/n)(W(it/n) - W((i-1)t/n)) \right] \\
= \lim_{n \to \infty} \sum_{i=1}^{n} 2\mathbb{E} \left[ (W((i-1)t/n) - W(0))[W(it/n) - W((i-1)t/n)] \right] \\
= 0.
\]

(Note the assumption that the limit and the expectation can be interchanged!)

But the mean of \( Y(t) = W(t)^2 \) is \( t \) which is definitely not zero! The two stochastic processes don’t agree even in the mean, so something is not right! If we agree that the integral definition and limit processes should be preserved, then the rules of calculus will have to change.

We can see how the rules of calculus must change by rearranging the summation. Use the simple algebraic identity

\[
2b(a - b) = (a^2 - b^2 - (a - b)^2)
\]
to re-write

\[
\int_0^t 2W(t) \, dW(t) = \lim_{n \to \infty} \sum_{i=1}^{n} 2W((i-1)t/n)[W(it/n) - W((i-1)t/n)] \\
= \lim_{n \to \infty} \sum_{i=1}^{n} (W(it/n)^2 - W((i-1)t/n)^2 - (W(it/n) - W((i-1)t/n))^2) \\
= \lim_{n \to \infty} \left( W(t)^2 - W(0)^2 - \sum_{i=1}^{n} (W(it/n) - W((i-1)t/n))^2 \right) \\
= W(t)^2 - \lim_{n \to \infty} \sum_{i=1}^{n} (W(it/n) - W((i-1)t/n))^2
\]

We recognize the second term in the last expression as being the quadratic variation of Wiener process, which we have already evaluated, and so

\[
\int_0^t 2W(t) \, dW(t) = W(t)^2 - t.
\]

**Itô’s Formula and Itô calculus**

Itô’s formula is an expansion expressing a stochastic process in terms of the deterministic differential and the Wiener process differential, that is, the stochastic differential equation for the process.

**Theorem 1** (Itô’s formula). If \( Y(t) \) is scaled Wiener process with drift, satisfying \( dY = r \, dt + \sigma \, dW \) and \( f \) is a twice continuously differentiable function, then \( Z(t) = f(Y(t)) \) is also a stochastic process satisfying the stochastic differential equation

\[
dZ = (r f'(Y) + (\sigma^2/2)f''(Y)) \, dt + (\sigma f'(Y)) \, dW
\]

In words, Itô’s formula in this form tells us how to expand (in analogy with the chain rule or Taylor’s formula) the differential of a process which is defined as an elementary function of scaled Brownian motion with drift.

*Itô’s formula* is often also called *Itô’s lemma* by other authors and texts. Most authors believe that this result is more important than a mere lemma, and so the text adopts the alternative name of “formula”. “Formula” also emphasizes the analogy with the chain “rule” and the Taylor “expansion”.
Example. Consider $Z(t) = W(t)^2$. Here the stochastic process is standard Brownian Motion, so $r = 0$ and $\sigma = 1$ so $dY = dW$. The twice continuously differentiable function $f$ is the squaring function, $f(x) = x^2$, $f'(x) = 2x$ and $f''(x) = 2$. Then according to Itô’s formula:

$$d(W^2) = (0 \cdot (2W(t)) + (1/2)(2)) \, dt + (1 \cdot 2W(t)) \, dW = dt + 2W(t) \, dW$$

Notice the additional $dt$ term! Note also that if we repeated the integration steps above in the example, we would obtain $W(t)^2$ as expected!

Example. Consider geometric Brownian motion $\exp(rt + \sigma W(t))$.

What SDE does geometric Brownian motion follow? Take $Y(t) = rt + \sigma W(t)$, so that $dY = r \, dt + \sigma \, dW$. Then geometric Brownian motion can be written as $Z(t) = \exp(Y(t))$, so $f$ is the exponential function. Itô’s formula is

$$dZ = (rf'(Y(t)) + (1/2)\sigma^2 f''(Y(t))) \, dt + \sigma f'(Y) \, dW$$

Computing the derivative of the exponential function and evaluating, $f'(Y(t)) = \exp(Y(t)) = Z(t)$ and likewise for the second derivative. Hence

$$dZ = (r + (1/2)\sigma^2)Z(t) \, dt + \sigma Z(t) \, dW$$

The case where $dY = dW$, that is the base process is standard Brownian Motion so $Z = f(W)$, occurs commonly enough that we record Itô’s formula for this special case:

**Corollary 1** (Itô’s Formula applied to functions of standard Brownian Motion). If $f$ is a twice continuously differentiable function, then $Z(t) = f(W(t))$ is also a stochastic process satisfying the stochastic differential equation

$$dZ = df(W) = (1/2)f''(W) \, dt + f'(W) \, dW.$$

**Guessing Processes from SDEs with Itô’s Formula**

One of the key needs we will have is to go in the opposite direction and convert SDEs to processes, in other words to solve SDEs. We take guidance from ordinary differential equations, where finding solutions to differential equations comes from judicious guessing based on a thorough understanding and
familiarity with the chain rule. For SDEs the solution depends on inspired guesses based on a thorough understanding of the formulas of stochastic calculus. Following the guess we require a proof that the proposed solution is an actual solution, again using the formulas of stochastic calculus.

A few rare examples of SDEs can be solved with explicit familiar functions. This is just like ODEs in that the solutions of many simple differential equations cannot be solved in terms of elementary functions. The solutions of the differential equations define new functions which are useful in applications. Likewise, the solution of an SDE gives us a way of defining new processes which are useful.

Example. Suppose we are asked to solve the SDE

$$dZ(t) = \sigma Z(t) \, dW .$$

We need an inspired guess, so we try

$$\exp(rt + \sigma W(t))$$

where $r$ is a constant to be determined while the $\sigma$ term is given in the SDE. Itô’s formula for the guess is

$$dZ = (r + (1/2)\sigma^2)Z(t) \, dt + \sigma Z(t) \, dW .$$

We notice that the stochastic term (or Wiener process differential term) is the same as the SDE. We need to choose the constant $r$ appropriately to eliminate the deterministic or drift differential term. If we choose $r$ to be $-(1/2)\sigma^2$ then the drift term in the differential equation would match the SDE we have to solve as well. We therefore guess

$$Y(t) = \exp(\sigma W(t) - (1/2)\sigma^2 t).$$

We should double check by applying Itô’s formula.

Soluble SDEs are scarce, and this one is special enough to give a name. It is the Doléan’s exponential of Brownian motion.

Sources

This discussion is adapted from Financial Calculus: An introduction to derivative pricing by M Baxter, and A. Rennie, Cambridge University Press,

Problems to Work for Understanding

1. Find the solution of the stochastic differential equation
   \[ dY(t) = Y(t) \, dt + 2Y(t) \, dW. \]

2. Find the solution of the stochastic differential equation
   \[ dY(t) = tY(t) \, dt + 2Y(t) \, dW. \]
   Note the difference with the previous problem, now the multiplier of the \( dt \) term is a function of time.

3. Find the solution of the stochastic differential equation
   \[ dY(t) = \mu Y(t) \, dt + \sigma Y(t) \, dW. \]

4. Find the solution of the stochastic differential equation
   \[ dY(t) = \mu t Y(t) \, dt + \sigma Y(t) \, dW. \]
   Note the difference with the previous problem, now the multiplier of the \( dt \) term is a function of time.

5. Find the solution of the stochastic differential equation
   \[ dY(t) = \mu(t) Y(t) \, dt + \sigma Y(t) \, dW. \]
   Note the difference with the previous problem, now the multiplier of the \( dt \) term is a general (technically, a locally bounded integrable) function of time.
Reading Suggestion:


Outside Readings and Links:

I check all the information on each page for correctness and typographical errors. Nevertheless, some errors may occur and I would be grateful if you would alert me to such errors. I make every reasonable effort to present current and accurate information for public use, however I do not guarantee the accuracy or timeliness of information on this website. Your use of the information from this website is strictly voluntary and at your risk.

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Steve Dunbar’s Home Page, http://www.math.unl.edu/~sdunbar1