Stochastic Processes and
Advanced Mathematical Finance

Path Properties of Brownian Motion

Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.
Section Starter Question

Provide an example of a continuous function which is not differentiable at some point. Why does the function fail to have a derivative at that point? What are the possible reasons that a derivative could fail to exist at some point?

Key Concepts

1. With probability 1 a Brownian Motion path is continuous but nowhere differentiable.

2. Although a Brownian Motion path is continuous, it has many counterintuitive properties not usually associated with continuous functions.

Vocabulary

1. Probability theory uses the term almost surely to indicate an event which occurs with probability 1. The complementary events occurring with probability 0 are sometimes called negligible events. In infinite sample spaces, it is possible to have meaningful events with probability zero. So to say an event occurs “almost surely” or is an negligible event is not an empty phrase.
Non-differentiability of Brownian Motion paths

Probability theory uses the term almost surely to indicate an event which occurs with probability 1. The complementary events occurring with probability 0 are sometimes called negligible events. In infinite sample spaces, it is possible to have meaningful events with probability zero. So to say an event occurs “almost surely” or is a negligible event is not an empty phrase.

**Theorem 1.** With probability 1 (i.e. almost surely) Brownian Motion paths are continuous functions.

To state this as a theorem may seem strange in view of property 4 of the definition of Brownian motion. Property 4 requires that Brownian motion is continuous. However, some authors weaken property 4 in the definition to only require that Brownian motion be continuous at \( t = 0 \). Then this theorem shows that the weaker definition implies the stronger definition used in this text. This theorem is difficult to prove, and well beyond the scope of this course. In fact, even the statement above is imprecise. Specifically, there is an explicit representation of the defining properties of Brownian Motion as a random variable \( W(t, \omega) \) which is a continuous function of \( t \) with probability 1. We need the continuity for much of what we do later, and so this theorem is stated here as a fact without proof.

**Theorem 2.** With probability 1 (i.e. almost surely) a Brownian Motion is nowhere (except possibly on set of Lebesgue measure 0) differentiable.

This property is even deeper and requires more understanding of analysis to prove than does the continuity theorem, so we will not prove it here. Rather, we use this fact as another piece of evidence of the strangeness of Brownian Motion.

In spite of one’s intuition from calculus, Theorem 2 shows that continuous, nowhere differentiable functions are actually common. Indeed, continuous,
nowhere differentiable functions are useful for stochastic processes. One can imagine non-differentiability by considering the function \( f(t) = |t| \) which is continuous but not differentiable at \( t = 0 \). Because of the corner at \( t = 0 \), the left and right limits of the difference quotient exist but are not equal. Even more to the point, the function \( t^{2/3} \) is continuous but not differentiable at \( t = 0 \) because of a sharp “cusp” there. The left and right limits of the difference quotient do not exist (more precisely, each approaches \( \pm \infty \)) at \( x = 0 \). One can imagine Brownian Motion as being spiky with tiny cusps and corners at every point. This becomes somewhat easier to imagine by thinking of the limiting approximation of Brownian Motion by scaled random walks. The re-scaled coin-flipping fortune graphs look spiky with many corners. The approximating graphs suggest why the theorem is true, although this is not sufficient for the proof.

Properties of the Path of Brownian Motion

**Theorem 3.** With probability 1 (i.e. almost surely) a Brownian Motion path has no intervals of monotonicity. That is, there is no interval \([a, b]\) with \( W(t_2) - W(t_1) > 0 \) (or \( W(t_2) - W(t_1) < 0 \)) for all \( t_2, t_1 \in [a, b] \) with \( t_2 > t_1 \)

**Theorem 4.** With probability 1 (i.e. almost surely) Brownian Motion \( W(t) \) has

\[
\limsup_{n \to \infty} \frac{W(n)}{\sqrt{n}} = +\infty, \\
\liminf_{n \to \infty} \frac{W(n)}{\sqrt{n}} = -\infty.
\]

From Theorem 4 and the continuity we can deduce that for arbitrarily large \( t_1 \), there is a \( t_2 > t_1 \) such that \( W(t_2) = 0 \). That is, Brownian Motion paths always cross the time-axis at some time greater than any arbitrarily large value of \( t \). Equivalently, Brownian Motion never eventually stays in the upper half-plane (or lower half-plane).

**Theorem 5.** With probability 1 (i.e. almost surely), 0 is an accumulation point of the zeros of \( W(t) \).

From Theorem 4 and the inversion \( tW(1/t) \) also being a standard Brownian motion, we heuristically deduce that 0 is an accumulation point of the
zeros of $W(t)$. That is, Standard Brownian Motion crosses the time axis arbitrarily often near 0.

**Theorem 6.** With probability 1 (i.e. almost surely) the zero set of Brownian Motion

$$\{t \in [0, \infty) : W(t) = 0\}$$

is an uncountable closed set with no isolated points.

**Theorem 7.** With probability 1 (i.e. almost surely) the graph of a Brownian Motion path has Hausdorff dimension $3/2$.

Roughly, this means that the graph of a Brownian Motion path is “fuzzier” or “thicker” than the graph of, for example, a continuously differentiable function which would have Hausdorff dimension 1. In popular language, this theorem says that Brownian Motion is a fractal.

**Sources**

This section is adapted from *Notes on Brownian Motion* by Yuval Peres, University of California Berkeley, Department of Statistics.

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**Algorithms, Scripts, Simulations**

**Algorithm**

For a given value of $p$ and number of steps $N$ on a time interval $[0, T]$ create a scaled random walk $\hat{W}_N(t)$ on $[0, T]$. Then for a given minimum increment $h_0$ up to a maximum increment $h_1$ create a sequence of equally spaced increments. Then at a fixed base-point, calculate the difference quotient for each of the increments. When plotting is available, plot the difference quotients versus the increments on a semi-logarithmic set of axes.

Because the difference quotients are computed using the scaled random walk approximation of the Wiener process, the largest possible slope is

$$\sqrt{T/N/(T/N)} = \sqrt{N/T}.$$
So the plotted difference quotients will “max out” once the increment is less than the scaled random walk step size.

The GeoGebra simulation finds the closest scaled random walk node less than the basepoint \( x_0 \). Then using the slider to set the value of \( h \), the GeoGebra simulation finds the closest scaled random node greater than \( x_0 + h \). The GeoGebra simulation draws the secant line through the two nodes to show that the difference quotients do not appear to converge as \( h \) decreases. The ultimate secant line for the smallest value of \( h \) is between two adjacent nodes and has a slope of \( \sqrt{N/T} \).

**GeoGebra**

**R** script for.

```R
p <- 0.5
N <- 1000
T <- 1
S <- array(0, c(N+1))
rw <- cumsum(2 * (runif(N) <= p) - 1)
S[2:(N+1)] <- rw

WcaretN <- function(x) {
  Delta <- T/N
  # add 1 since arrays are 1-based
  prior = floor(x/Delta) + 1
  subsequent = ceiling(x/Delta) + 1
  retval <- sqrt(Delta) * (S[prior] + ((x/Delta+1) - prior) * (S[subsequent] - S[prior]))
}

h0 <- 1e-7
h1 <- 1e-2
m = 30
basepoint = 0.5

h <- seq(h0, h1, length=m)
x0 <- array(basepoint, c(m))
diffquotients <- abs(WcaretN(x0 + h) - WcaretN(x0))/h
```

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p = 0.5;
global N = 1000;
global T = 1;

global S
S = zeros(N+1, 1);
S(2:N+1) = cumsum( 2 * (rand(N,1)<=p) - 1);

function retval = WcaretN(x)
global N;
global T;
global S;
step = T/N;

    # add 1 since arrays are 1-based
prior = floor(x/step) + 1;
subsequent = ceil(x/step) + 1;
    retval = sqrt(step)*((S(prior) + ((x/step+1) - prior).*((S(subsequent)-S(prior))));
endfunction

h0 = 1e-7;
h1 = 1e-2;
m = 30;
basepoint = 0.5;

h = transpose(linspace(h0,h1, m));
x0 = basepoint * ones(m,1);

diffquotients = abs( WcaretN( x0 + h) - WcaretN(x0) ) ./ h

semilogy( h, diffquotients)
xlabel("h")
Perl PDL script for

```perl
$p = 0.5;
$N = 1000;
$T = 1;

# the random walk
$S = zeros($N + 1);
$S (1 : $N) .= cumsumover(2 * (random($N) <= $p) - 1);

# function \( W^{\alpha} \) interpolating random walk
sub WcaretN {
    my $x = shift @_;  
    $Delta = $T / $N;
    $prior = floor($x / $Delta);
    $subsequent = ceil($x / $Delta);
    $retval =
        sqrt($Delta) * (
            $S($prior)
            + ( ($x / $Delta) - $prior )
            * ($S($subsequent) - $S($prior))
        );
}

$h0 = 1e-7;
$h1 = 1e-2;
$m  = 30;
$basepoint = 0.5;

$h = zeroes($m)->xlinvals($h0, $h1);
$x0 = $basepoint * ones($m);

$diffquotients = abs(WcaretN($x0 + $h) - WcaretN($x0)) / $h;

# file output to use with external plotting programming
# such as gnuplot, R, octave, etc.
# Start gnuplot, then from gnuplot prompt
# set logscale y
```

\[
ylabel("abs(W(x_0+h) - W(x_0))/h)"
\]
# plot "pathproperties.dat" with lines

open( F, ">pathproperties.dat") || die "cannot write: "$!");

foreach $j (0 .. $m - 1) {
    print F$h->range( [$j] ), " ", $diffquotients->range( [$j] ), "\n";
}
close(F);

SciPy

```python
import scipy

p = 0.5
N = 1000
T = 1.

# the random walk
S = scipy.zeros(N+1)
S[1:N+1] = scipy.cumsum(2*(scipy.random.random(N) <= p) - 1)

def WcaretN(x):
    Delta = T/N
    prior = scipy.floor(x/Delta).astype(int)
    subsequent = scipy.ceil(x/Delta).astype(int)
    return scipy.sqrt(Delta)*(S[prior] + (x/Delta - prior))*(S[subsequent] - S[prior])

h0 = 1e-7
h1 = 1e-2
m = 30
basepoint = 0.5

h = scipy.linspace(h0, h1, m)
x0 = basepoint * scipy.ones(30)

diffquotients = scipy.absolute(WcaretN(x0 + h) - WcaretN(x0))/h
```

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# optional file output to use with external plotting programming
# such as gnuplot, R, octave, etc.
# Start gnuplot, then from gnuplot prompt
# set logscale y
# plot "pathproperties.dat" with lines
f = open('pathproperties.dat', 'w')
for j in range(0, m-1):
    f.write(str(h[j])+' '+str(diffquotients[j])+'
');
f.close()

Problems to Work for Understanding

1. In an infinite sequence of fair coin flips, consider the event that there are only finitely many tails. What is the probability of this event? Is this event empty? Is this event impossible?

2. Provide a more complete heuristic argument based on Theorem 4 that almost surely there is a sequence $t_n$ with $\lim_{t \to \infty} t_n = \infty$ such that $W(t) = 0$

3. Provide a heuristic argument based on Theorem 5 and the shifting property that the zero set of Brownian Motion

$$\{t \in [0, \infty) : W(t) = 0\}$$

has no isolated points.

4. Looking in more advanced references, find another property of Brownian Motion which illustrates strange path properties.
Reading Suggestion:

References


Outside Readings and Links:

1.

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