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Math 489/Math 889
Stochastic Processes and
Advanced Mathematical Finance
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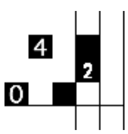
Path Properties of Brownian Motion



Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.

Question of the Day

Provide an example of a continuous function which is not differentiable at some point. Why does the function fail to have a derivative at that point? What are the possible reasons that a derivative could fail to exist at some point?

Key Concepts

1. With probability 1 a Brownian Motion path is continuous but *nowhere* differentiable.

Vocabulary

1. In probability theory, the term **almost surely** is used to indicate an event which occurs with probability 1. In infinite sample spaces, it is possible to have meaningful events with probability zero. So to say an event occurs "almost surely" is not an empty phrase. Events occurring with probability zero are sometimes called **negligible events**.

Mathematical Ideas

Properties of the Path of Brownian Motion

Theorem 1. *With probability* 1 *(i.e. almost surely) Brownian Motion paths are continuous functions.*

To state this as a theorem may seem strange in view of property 4 of the definition of Brownian motion. Property 4 requires that Brownian motion is continuous. However, some authors weaken property 4 in the definition to only require that Brownian motion be continuous at $t = 0$. Then this theorem shows that the weaker definition implies the stronger definition used in this text. This theorem is difficult to prove, and well beyond the scope of this course. In fact, even the statement above is imprecise. Specifically, there is an explicit representation of the defining properties of Brownian Motion as a function in which (with probability 1) $W(t, \omega)$ is a continuous function of t. We need the continuity for much of what we do later, and so this theorem is stated here again as a fact without proof.

Theorem 2. *With probability* 1 *(i.e. almost surely) a Brownian Motion is nowhere (except possibly on set of Lebesgue measure* 0*) differentiable.*

This property is even deeper and requires more machinery to prove than does the continuity theorem, so we will not prove it here. Rather, we use this fact as another piece of evidence of the strangeness of Brownian Motion.

In spite of one's intuition from calculus, Theorem 2 shows that continuous, nowhere differentiable functions are actually common. Indeed, continuous, nowhere differentiable functions are useful for stochastic processes. One can imagine non-differentiability by considering the function $f(t) = |t|$ which is continuous but not differentiable at $t = 0$. Because of the corner at $t = 0$, the left and right limits of the difference quotient exist but are not equal. Even more to the point, the function $t^{2/3}$ is continuous but not differentiable at $t = 0$ because of a sharp "cusp" there. The left and right limits of the difference quotient do not exist (more precisely, they approach $\pm\infty$) at $x = 0$. One can imagine Brownian Motion as being spiky with tiny cusps and corners at every point. This becomes somewhat easier to imagine by thinking of the limiting approximation of Brownian Motion by coin-flipping fortunes. The re-scaled coin-flipping fortune graphs look spiky with corners everywhere. The approximating graphs suggest why the theorem is true, although this is not sufficient for the proof.

Theorem 3. *With probability* 1 *(i.e. almost surely) a Brownian Motion path has no intervals of monotonicity. That is, there is no interval* $[a, b]$ *with* $W(t_2) - W(t_1) > 0$ *(or* $W(t_2) - W(t_1) < 0$*) for all* $t_2, t_1 \in [a, b]$ *with* $t_2 > t_1$

Theorem 4. *With probability 1 (i.e. almost surely) Brownian Motion* $W(t)$ *has*

$$\limsup_{n\to\infty} \frac{W(n)}{\sqrt{n}} = +\infty,$$
$$\liminf_{n\to\infty} \frac{W(n)}{\sqrt{n}} = -\infty.$$

From Theorem 4 and the continuity we can deduce that for arbitrarily large t_1, there is a $t_2 > t_1$ such that $W(t_2) = 0$. That is, Brownian Motion paths cross the time-axis at some time greater than any arbitrarily large value of t.

Theorem 5. *With probability 1 (i.e. almost surely),* 0 *is an accumulation point of the zeros of* $W(t)$*.*

From Theorem 4 and the inversion $tW(1/t)$ also being a standard Brownian motion, we deduce that 0 is an accumulation point of the zeros of $W(t)$. That is, Standard Brownian Motion crosses the time axis arbitrarily near 0.

Theorem 6. *With probability 1 (i.e. almost surely) the zero set of Brownian Motion*

$$\{t \in [0, \infty) : W(t) = 0\}$$

is an uncountable closed set with no isolated points.

Theorem 7. *With probability* 1 *(i.e. almost surely) the graph of a Brownian Motion path has Hausdorff dimension* $3/2$*.*

This means that the graph of a Brownian Motion path is “fuzzier” or “thicker” than the graph of, for example, a continuously differentiable function which would have Hausdorff dimension 1.

Sources

This section is adapted from: Notes on Brownian Motion. by Yuval Peres, University of California Berkeley, Department of Statistics.

Problems to Work for Understanding

1. Provide a more complete heuristic argument based on Theorem 4 that almost surely there is a sequence t_n with $\lim_{t \to \infty} t_n = \infty$ such that $W(t) = 0$

2. Provide a heuristic argument based on Theorem 5 and the shifting property that the zero set of Brownian Motion

$$\{t \in [0, \infty) : W(t) = 0\}$$

   has no isolated points.

3. Looking in more advanced references, find another property of Brownian Motion which illustrates strange path properties.

Reading Suggestion:

References

[1] David Freedman. *Brownian Motion and Diffusions.* Holden-Day, 1971. QA274.75F74.

[2] I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus*. Graduate Texts in Mathematics. Springer Verlag, second edition edition, 1997.

[3] S. Karlin and H. Taylor. *A Second Course in Stochastic Processes*. Academic Press, 1981.

[4] Steven E. Shreve. *Stochastic Calculus For Finance*, volume Volume II of *Springer Finance*. Springer Verlag, 2004.

[5] Steven E. Shreve. *Stochastic Calculus For Finance*, volume Volume I of *Springer Finance*. Springer Verlag, 2004.

Outside Readings and Links:

1. Notes on Brownian Motion. Yuval Peres, University of California Berkeley, Department of Statistics

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