Stochastic Processes and
Advanced Mathematical Finance

Hitting Times and Ruin Probabilities

Rating
Mathematically Mature: may contain mathematics beyond calculus with proofs.
Section Starter Question

What is the probability that a simple random walk with $p = 1/2 = q$ starting at the origin will hit value $a > 0$ before it hits value $-b < 0$, where $b > 0$? What do you expect in analogy for the standard Wiener process and why?

Key Concepts

1. With the Reflection Principle, we can derive the p.d.f of the hitting time $T_a$.

2. With the hitting time, we can derive the c.d.f. of the maximum of the Wiener Process on the interval $0 \leq u \leq t$.

Vocabulary

1. The **Reflection Principle** for the Wiener process reflected about a first passage has the same distribution as the original motion.

2. The **hitting time** $T_a$ is the first time the Wiener process assumes the value $a$. In notation from analysis

$$T_a = \inf\{t > 0 : W(t) = a\}.$$
Mathematical Ideas

Hitting Times

Consider the standard Wiener process \( W(t) \), which starts at \( W(0) = 0 \). Let \( a > 0 \). The hitting time \( T_a \) is the first time the Wiener process hits \( a \). In notation from analysis

\[
T_a = \inf\{t > 0 : W(t) = a\}.
\]

Note the very strong analogy with the duration of the game in the gambler’s ruin.

Some Wiener process sample paths will hit \( a > 0 \) fairly directly. Others will make an excursion to negative values and take a long time to finally reach \( a \). Thus \( T_a \) will have a probability distribution. We determine that probability distribution by a heuristic procedure similar to the first step analysis we made for coin-flipping fortunes.

Specifically, we consider a probability by conditioning, that is, conditioning on whether or not \( T_a \leq t \), for some given value of \( t \).

\[
P[ W(t) \geq a ] = P[ W(t) \geq a | T_a \leq t ] P[ T_a \leq t ] + P[ W(t) \geq a | T_a > t ] P[ T_a > t ]
\]

Now note that the second conditional probability is 0 because it is an empty event. Therefore:

\[
P[ W(t) \geq a ] = P[ W(t) \geq a | T_a \leq t ] P[ T_a \leq t ].
\]

Now, consider Wiener process “started over” again at the time \( T_a \) when it hits \( a \). By the shifting transformation from the previous section, the “started-over” process has the distribution of Wiener process again, so

\[
P[ W(t) \geq a | T_a \leq t ] = P[ W(t) \geq a | W(T_a) = a, T_a \leq t ]
\]
\[
= P[ W(t) - W(T_a) \geq 0 | T_a \leq t ]
\]
\[
= 1/2.
\]

This argument is a specific example of the Reflection Principle for the Wiener process. It says that the Wiener process reflected about a first passage has the same distribution as the original motion.

Thus

\[
P[ W(t) \geq a ] = (1/2)P[ T_a \leq t ].
\]
or

\[ \mathbb{P} [T_{a} \leq t] = 2\mathbb{P} [W(t) \geq a] \]
\[ = \frac{2}{\sqrt{2\pi t}} \int_{a}^{\infty} \exp\left(-\frac{u^2}{2t}\right) \, du \]
\[ = \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} \exp\left(-\frac{v^2}{2}\right) \, dv \]

(note the change of variables \( v = u/\sqrt{t} \) in the second integral) and so we have derived the c.d.f. of the hitting time random variable. One can easily differentiate to obtain the p.d.f

\[ f_{T_{a}}(t) = \frac{a}{\sqrt{2\pi t}} t^{-3/2} \exp\left(-\frac{a^2}{2t}\right). \]

Actually, this argument contains a serious logical gap, since \( T_{a} \) is a random time, not a fixed time. That is, the value of \( T_{a} \) is different for each sample path, it varies with \( \omega \). On the other hand, the shifting transformation defined in the prior section depends on having a fixed time, called \( h \) in that section. To fix this logical gap, we must make sure that “random times” act like fixed times. Under special conditions, random times can act like fixed times. Specifically, this proof can be fixed and made completely rigorous by showing that the standard Wiener process has the strong Markov property and that \( T_{a} \) is a Markov time corresponding to the event of first passage from 0 to \( a \).

Note that deriving the p.d.f. of the hitting time is much stronger than the analogous result for the duration of the game until ruin in the coin-flipping game. There we were only able to derive an expression for the expected value of the hitting time, not the probability distribution of the hitting time. Now we are able to derive the probability distribution of the hitting time fairly intuitively (although strictly speaking there is a gap). Here is a place where it is simpler to derive a quantity for Wiener process than it is to derive the corresponding quantity for random walk.

Let us now consider the probability that the Wiener process hits \( a > 0 \), before hitting \(-b < 0\), where \( b > 0 \). To compute this we will make use of the interpretation of Standard Wiener process as being the limit of the symmetric random walk. Recall from the exercises following the section on the gambler’s ruin in the fair \((p = 1/2 = q)\) coin-flipping game that the
probability that the random walk goes up to value \( a \) before going down to value \( b \) when the step size is \( \Delta x \) is

\[
P[ \text{to } a \text{ before } -b ] = \frac{b \Delta x}{(a + b) \Delta x} = \frac{b}{a + b}
\]

Thus, the probability of hitting \( a > 0 \) before hitting \( -b < 0 \) does not depend on the step size, and also does not depend on the time interval. Therefore, passing to the limit in the scaling process for random walks, the probabilities should remain the same. Here is a place where it is easier to derive the result from the coin-flipping game and pass to the limit than to derive the result directly from Wiener process principles.

The Distribution of the Maximum

Let \( t \) be a given time, let \( a > 0 \) be a given value, then

\[
P\left[ \max_{0 \leq u \leq t} W(u) \geq a \right] = P[T_a \leq t]
\]

\[
= \frac{2}{\sqrt{2\pi}} \int_{a/\sqrt{t}}^{\infty} \exp(-y^2/2) \, dy
\]

Sources

Algorithms, Scripts, Simulations

Algorithm

Set a time interval length $T$ sufficiently long to have a good chance of hitting the fixed value $a$ before fixed time $t < T$. Set a large value $n$ for the length of the random walk used to create the Wiener process, then fill an $n \times k$ matrix with Bernoulli random variables. Cumulatively sum the Bernoulli random variables to create a scaled random walk approximating the Wiener process. For each random walk, find when the hitting time is encountered. Also find the maximum value of the scaled random walk on the interval $[0, t]$. Since the approximation is piecewise linear, only the nodes need to be examined. Compute the fraction of the $k$ walks which have a hitting time less than $t$ or a maximum greater than $a$ on the interval $[0, t]$. Compare the fraction to the theoretical value.

**Technical Note:** The careful reader will note that the hitting time $T_a = \inf\{t > 0 : W(t) = a\}$ and the events $[T_a \leq t]$ and $[\max_{0 \leq u \leq t} W(u) \geq a]$ are “global” events on the set of all Wiener process paths. However, the definition of Wiener process only has prescribed probability distributions on the values at specified times. Implicit in the definition of the Wiener process is a probability distribution on the “global” set of Wiener processes, but the proof of the existence of the probability distribution is beyond the scope of this text. Moreover, there is no easy distribution function to compute the probability of such events as there is with the normal probability distribution.

The algorithm approximates the probability by counting how many of $k$ scaled binomial random variables hit a value greater than or equal to $a$ at a node which corresponds to a time less than or equal to $t$. Convergence of the counting probability to the global Wiener process probability requires justification with *Donsker’s Invariance Principle*. The principle says that the piecewise linear processes $\hat{W}_N(t)$ converge in distribution to the Wiener process.

To apply Donsker’s Principle, consider the functional

$$\phi(f) = h(\max_{0 \leq u \leq t} f(u))$$

where $h(\cdot)$ is a bounded continuous function. The convergence in distribution from the Invariance Principle implies that

$$\mathbb{E} \left[ \phi(\hat{W}_N(t)) \right] \rightarrow \mathbb{E} [\phi W].$$
Now taking $h(\cdot)$ to be an approximate indicator function on the interval $[a, \infty)$ shows that the counting probability converges to the global Wiener process probability. The details require careful analysis.

**Geogebra GeoGebra**

**R**

R script for hitting time.

```r
T <- 10
a <- 1
time <- 2
p <- 0.5
n <- 10000
k <- 1000
Delta = T/n

winLose <- 2 * (array( 0+(runif(n*k) <= p), dim=c(n,k))) - 1
# 0+ coerces Boolean to numeric
totals <- apply( winLose, 2, cumsum)

paths <- array( 0 , dim=c(n+1 , k) )
paths[2:(n+1) , 1:k] <- sqrt( Delta )*totals

hitIndex <- apply( 0+(paths <= a), 2, (function(x) match(0 , x, nomatch =n+2)) )
# If no hitting on a walk, nomatch=n+2 sets the hitting
# time to be two more than the number of steps, one more
# than
# the column length. Without the nomatch option, get NA
# which
# works poorly with the comparison

hittingTime = Delta*(hitIndex -1)
## subtract 1 since vectors are 1-based

probHitlessTa <- sum( 0+(hittingTime <= time))/k
probMax = sum( 0+( apply(paths[1:((time/Delta)+1)], , 2, max) >= a ) )/k
theoreticalProb = 2*pnorm(a/sqrt(time), lower=FALSE)

cat(sprintf("Empirical probability Wiener process paths hit %f before %f: %f \n", a, time, probHitlessTa ))
```

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Octave script for hitting time.

```octave
T = 10;
a = 1;
time = 2;
p = 0.5;
n = 10000;
k = 1000;
Delta = T/n;

winLose = 2 * (rand(n,k) <= p) - 1;
# -1 for Tails, 1 for Heads
totals = cumsum(winLose);
# -n..n (every other integer) binomial rv sample

paths = sqrt(Delta)*[zeros(1,k); totals];

hittingTime = zeros(1,k);
for j = 1:k
    hitIndex = find(paths(:,j) >= a);
    if ( !rows(hitIndex) )
        hittingTime(j) = Delta * (n+2);
        ## If no hitting on a walk, set hitting time to be two
        ## more than the number of steps, one more than the
        ## column length.
    else
        hittingTime(j) = Delta * (hitIndex(1)-1);
        ## some hitting time
        ## subtract 1 since vectors are 1-based
    endif
endfor

probHitlessTa = sum( (hittingTime <= time) ) /k;
probMax = sum( max(paths(1:(time/Delta+1),:)) >= a)/k;
theoreticalProb = 2*(1 - normcdf(a/sqrt(time)));
```

```
cat(sprintf("Empirical probability Wiener process paths
greater than %f before %f: %f \n", a, time, probMax ))
cat(sprintf("Theoretical probability: %f \n", theoreticalProb ))
```
printf("Empirical probability Wiener process paths hit %f
before %f: %f \n", a, time, probHitlessTa )
printf("Empirical probability Wiener process paths
greater than %f before %f: %f \n", a, time, probMax )
printf("Theoretical probability: %f \n", theoreticalProb)

Perl  Perl PDL script for hitting time

use PDL::NiceSlice;

sub pnorm {
  my ( $x, $sigma, $mu ) = @_;
  $sigma = 1 unless defined($sigma);
  $mu = 0 unless defined($mu);

  return 0.5 * ( 1 + erf( ( $x - $mu ) / ( sqrt(2) * $sigma ) ) );
}

$T = 10;
$a = 1;
$time = 2;
$p = 0.5;
$n = 10000;
$k = 1000;

$Delta = $T / $n;

$winLose = 2 * ( random( $k, $n ) <= $p ) - 1;
$totals = ( cumusumover $winLose->xchg( 0, 1 ) )->transpose;

$paths = zeroes( $k, $n + 1 );

# use PDL::NiceSlice on next line
$paths ( 0 : ( $k - 1 ), 1 : $n ) .= sqrt($Delta) * $totals;

$hita = $paths->setbadif( $paths <= $a );

$hitIndex =
SciPy | Scientific Python script for hitting time.

```python
import scipy

T = 10.0
# note type float
a = 1
time = 2

p = 0.5
n = 1000
k = 50

Delta = T/n

winLose = 2*(scipy.random.random((n,k)) <= p) - 1
totals = scipy.cumsum(winLose, axis = 0)

paths = scipy.zeros((n+1,k), dtype=float)
paths[1:n+1, :] = scipy.sqrt(Delta) * totals

def match(x,arry,nomatch=None):
```
```python
if arry[scipy.where(arry >= x)].any():
    return scipy.where(arry >= x)[0][0] - 1
else:
    return nomatch

# arguments: x is a scalar, arry is a python list, value of nomatch is scalar
# returns the first index of first of its first argument in its second argument
# but if a is not there, returns the value nomatch
# modeled on the R function "match", but with less generality

hitIndex = scipy.apply_along_axis(lambda x: (match(a, x, nomatch=n+2)), 0, paths)
# If no ruin or victory on a walk, nomatch=n+2 sets the hitting
# time to be two more than the number of steps, one more than
# the column length.

hittingTime = Delta * hitIndex

probHitlessTa = (scipy.sum(hittingTime < time).astype('float'))/k
probMax = (scipy.sum(scipy.amax(paths[0:scipy.floor(time/Delta)+1, :], axis=0) >= a).astype('float'))/k
from scipy.stats import norm
theoreticalProb = 2 * (1 - norm.cdf(a/scipy.sqrt(time)))

print "Empirical probability Wiener process paths hit ", a, " before ", time, " is ", probHitlessTa
print "Empirical probability Wiener process paths greater than ", a, " before ", time, " is ", probMax
print "Theoretical probability:", theoreticalProb
```
Problems to Work for Understanding

1. Differentiate the c.d.f. of $T_a$ to obtain the expression for the p.d.f of $T_a$.

2. Show that $\mathbb{E}[T_a] = \infty$ for $a > 0$.

3. Suppose that the fluctuations of a share of stock of a certain company are well described by a Wiener process. Suppose that the company is bankrupt if ever the share price drops to zero. If the starting share price is $A(0) = 5$, what is the probability that the company is bankrupt by $t = 25$? What is the probability that the share price is above 10 at $t = 25$?

4. Suppose you own one share of stock whose price changes according to a Wiener process. Suppose you purchased the stock at a price $b + c$, $c > 0$ and the present price is $b$. You have decided to sell the stock either when it reaches the price $b + c$ or when an additional time $t$ goes by, whichever comes first. What is the probability that you do not recover your purchase price?

5. Modify the scripts by setting $p > 0.5$ or $p < 0.5$. What happens to the hitting time?

6. (a) Modify the scripts to plot the probability that the hitting time is less than or equal to $a$ as a function of $a$.

(b) Modify the scripts to plot the probability that the hitting time is less than or equal to $a$ as a function of $\text{end}$. On the same set of axes plot the theoretical probability as a function of $t$.

Reading Suggestion:

References


**Outside Readings and Links:**


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