The Definition of Brownian Motion and the Wiener Process

Rating
Mathematically Mature: may contain mathematics beyond calculus with proofs.
Section Starter Question

Some mathematical objects are defined by a formula or an expression. Some other mathematical objects are defined by their properties, not explicitly by an expression. That is, the objects are defined by how they act, not by what they are. Can you name a mathematical object defined by its properties?

Key Concepts

1. We define Brownian motion in terms of the normal distribution of the increments, the independence of the increments, the value at 0, and its continuity.

2. The joint density function for the value of Brownian motion at several times is a multivariate normal distribution.

Vocabulary

1. Brownian motion is the physical phenomenon named after the English botanist Robert Brown who discovered it in 1827. Brownian motion is the zig-zagging motion exhibited by a small particle, such as a grain of pollen, immersed in a liquid or a gas. Albert Einstein gave the first explanation of this phenomenon in 1905. He explained Brownian motion by assuming the immersed particle was constantly buffeted by the molecules of the surrounding medium. Since then the abstracted process has been used for modeling the stock market and in quantum mechanics.
2. The **Wiener process** is the mathematical definition and abstraction of the physical process as a stochastic process. The American mathematician Norbert Wiener gave the definition and properties in a series of papers starting in 1918. Generally, the terms **Brownian motion** and **Wiener process** are the same, although Brownian motion emphasizes the physical aspects and Wiener process emphasizes the mathematical aspects.

3. **Bachelier process** means the same thing as Brownian motion and Wiener process. In 1900, Louis Bachelier introduced the limit of random walk as a model for prices on the Paris stock exchange, and so is the originator of the mathematical idea now called Brownian motion. This term is occasionally found in financial literature.

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### Mathematical Ideas

#### Definition of Wiener Process

Previously we considered a *discrete time random process*. That is, at discrete times $n = 1, 2, 3, \ldots$ corresponding to coin flips, we considered a sequence of random variables $T_n$. We are now going to consider a *continuous time random process*, a function $W(t)$ that is a random variable at each time $t \geq 0$. To say $W(t)$ is a random variable at each time is too general so we must put more restrictions on our process to have something interesting to study.

*Definition* (Wiener Process). The **Standard Wiener Process** is a stochastic process $W(t)$, for $t \geq 0$, with the following properties:

1. Every increment $W(t) - W(s)$ over an interval of length $t - s$ is normally distributed with mean 0 and variance $t - s$, that is

   \[ W(t) - W(s) \sim N(0, t - s). \]
2. For every pair of disjoint time intervals \([t_1, t_2]\) and \([t_3, t_4]\), with \(t_1 < t_2 \leq t_3 < t_4\), the increments \(W(t_4) - W(t_3)\) and \(W(t_2) - W(t_1)\) are independent random variables with distributions given as in part 1, and similarly for \(n\) disjoint time intervals where \(n\) is an arbitrary positive integer.

3. \(W(0) = 0\).

4. \(W(t)\) is continuous for all \(t\).

Note that property 2 says that if we know \(W(s) = x_0\), then the independence (and \(W(0) = 0\)) tells us that further knowledge of the values of \(W(\tau)\) for \(\tau < s\) give no added knowledge of the probability law governing \(W(t) - W(s)\) with \(t > s\). More formally, this says that if \(0 \leq t_0 < t_1 < \ldots < t_n < t\), then

\[
\mathbb{P}[W(t) \geq x \mid W(t_0) = x_0, W(t_1) = x_1, \ldots, W(t_n) = x_n] = \mathbb{P}[W(t) \geq x \mid W(t_n) = x_n].
\]

This is a statement of the Markov property of the Wiener process.

Recall that the sum of independent random variables that are respectively normally distributed with mean \(\mu_1\) and \(\mu_2\) and variances \(\sigma_1^2\) and \(\sigma_2^2\) is a normally distributed random variable with mean \(\mu_1 + \mu_2\) and variance \(\sigma_1^2 + \sigma_2^2\), (see [Moment Generating Functions]). Therefore for increments \(W(t_3) - W(t_2)\) and \(W(t_2) - W(t_1)\) the sum \(W(t_3) - W(t_2) + W(t_2) - W(t_1) = W(t_3) - W(t_1)\) is normally distributed with mean 0 and variance \(t_3 - t_1\) as we expect. Property 2 of the definition is consistent with properties of normal random variables.

Let

\[
p(x, t) = \frac{1}{\sqrt{2\pi t}} \exp(-x^2/(2t))
\]

denote the probability density for a \(N(0, t)\) random variable. Then to derive the joint density of the event

\[
W(t_1) = x_1, W(t_2) = x_2, \ldots, W(t_n) = x_n
\]

with \(t_1 < t_2 < \ldots < t_n\), it is equivalent to know the joint probability density of the equivalent event

\[
W(t_1) - W(0) = x_1, W(t_2) - W(t_1) = x_2 - x_1, \ldots, W(t_n) - W(t_{n-1}) = x_n - x_{n-1}.
\]
Then by part 2, we immediately get the expression for the joint probability density function:

\[ f(x_1, t_1; x_2, t_2; \ldots; x_n, t_n) = p(x_1, t)p(x_2 - x_1, t_2 - t_1) \ldots p(x_n - x_{n-1}, t_n - t_{n-1}). \]

**Comments on Modeling Security Prices with the Wiener Process**

A plot of security prices over time and a plot of one-dimensional Brownian motion versus time has at least a superficial resemblance.

If we were to use Brownian motion to model security prices (ignoring for the moment that security prices are better modeled with the more sophisticated geometric Brownian motion rather than simple Brownian motion) we would need to verify that security prices have the 4 defining properties of Brownian motion:

1. The assumption of normal distribution of stock price changes seems to be a reasonable first assumption. Figure 2 illustrates this reasonable agreement. The Central Limit Theorem provides a reason to believe the agreement, assuming the requirements of the Central Limit Theorem are met, including independence. (Unfortunately, although the figure shows what appears to be reasonable agreement a more rigorous statistical analysis shows that the data distribution does not match normality.)

Another good reason for still using the assumption of normality for the increments is that the normal distribution is easy to work with. The normal probability density uses simple functions familiar from calculus, the normal cumulative probability distribution is tabulated, the moment-generating function of the normal distribution is easy to use, and the sum of independent normal distributions is again normal. A substitution of another distribution is possible but the resulting stochastic process models are difficult to analyze, beyond the scope of this model.

However, the assumption of a normal distribution ignores the small possibility that negative stock prices could result from a large negative
Figure 1: Graph of the Dow-Jones Industrial Average from February 17, 2015 to February 16, 2016 (blue line) and a random walk with normal increments with the same initial value and variance (red line).
Figure 2: A standardized density histogram of 1000 daily close-to-close returns on the S & P 500 Index, from February 29, 2012 to March 1, 2012, up to February 21, 2016 to February 22, 2016.
change. This is not reasonable. (The log normal distribution from geometric Brownian motion that avoids this possibility is a better model). Moreover, the assumption of a constant variance on different intervals of the same length is not a good assumption since stock volatility itself seems to be volatile. That is, the variance of a stock price changes and need not be proportional to the length of the time interval.

2. The assumption of independent increments seems to be a reasonable assumption, at least on a long enough term. From second to second, price increments are probably correlated. From day to day, price increments are probably independent. Of course, the assumption of independent increments in stock prices is the essence of what economists call the Efficient Market Hypothesis, or the Random Walk Hypothesis, which we take as a given in order to apply elementary probability theory.

3. The assumption of $W(0) = 0$ is simply a normalizing assumption and needs no discussion.

4. The assumption of continuity is a mathematical abstraction of the collected data, but it makes sense. Securities are traded second by second or minute by minute where prices jump discretely by small amounts. Examined on a scale of day by day or week by week, then the short-time changes are tiny and in comparison prices seem to change continuously.

At least as a first assumption, we will try to use Brownian motion as a model of stock price movements. Remember the mathematical modeling proverb quoted earlier: All mathematical models are wrong, some mathematical models are useful. The Brownian motion model of stock prices is at least moderately useful.

**Conditional Probabilities**

According to the defining property 1 of Brownian motion, we know that if $s < t$, then the conditional density of $X(t)$ given $X(s) = B$ is that of a normal random variable with mean $B$ and variance $t - s$. That is,

$$\mathbb{P}[X(t) \in (x, x + \Delta x) \mid X(s) = B] \approx \frac{1}{\sqrt{2\pi (t - s)}} \exp \left( -\frac{(x - B)^2}{2(t - s)} \right) \Delta x.$$
This gives the probability of Brownian motion being in the neighborhood of \( x \) at time \( t \), \( t - s \) time units into the future, given that Brownian motion is at \( B \) at time \( s \), the present.

However the conditional density of \( X(s) \) given that \( X(t) = B \), \( s < t \) is also of interest. Notice that this is a much different question, since \( s \) is “in the middle” between 0 where \( X(0) = 0 \) and \( t \) where \( X(t) = B \). That is, we seek the probability of being in the neighborhood of \( x \) at time \( s \), \( t - s \) time units in the past from the present value \( X(t) = B \).

**Theorem 1.** The conditional distribution of \( X(s) \), given \( X(t) = B \), \( s < t \), is normal with mean \( Bs/t \) and variance \((s/t)(t - s)\).

\[
\mathbb{P}[X(s) \in (x, x + \Delta x) \mid X(t) = B] \approx \frac{1}{\sqrt{2\pi(s/t)(t - s)}} \exp\left(\frac{-(x - Bs/t)^2}{2(t - s)}\right) \Delta x.
\]

**Proof.** The conditional density is

\[
f_{s \mid t}(x \mid B) = \frac{f_s(x) f_{t-s}(B - x)}{f_t(B)} = K_1 \exp\left(\frac{-x^2 - (B - x)^2}{2s - 2(t - s)}\right)
= K_2 \exp\left(-x^2\left(\frac{1}{2s} + \frac{1}{2(t - s)}\right) + \frac{Bx}{t - s}\right)
= K_2 \exp\left(-\frac{t}{2s(t - s)} \cdot x^2 - \frac{2sBx}{t}\right)
= K_3 \exp\left(\frac{-t(x - Bs/t)^2}{2s(t - s)}\right)
\]

where \( K_1, K_2, \) and \( K_3 \) are constants that do not depend on \( x \). For example, \( K_1 \) is the product of \( 1/\sqrt{2\pi s} \) from the \( f_s(x) \) term, and \( 1/\sqrt{2\pi(t - s)} \) from the \( f_{t-s}(B - x) \) term, times the \( 1/f_t(B) \) term in the denominator. The \( K_2 \) term multiplies in an \( \exp(-B^2/(2(t - s))) \) term. The \( K_3 \) term comes from the adjustment in the exponential to account for completing the square. We know that the result is a conditional density, so the \( K_3 \) factor must be the correct normalizing factor, and we recognize from the form that the result is a normal distribution with mean \( Bs/t \) and variance \((s/t)(t - s)\). \( \square \)
Corollary 1. The conditional density of $X(t)$ for $t_1 < t < t_2$ given $X(t_1) = A$ and $X(t_2) = B$ is a normal density with mean

$$A + \left(\frac{B - A}{t_2 - t_1}\right)(t - t_1)$$

and variance

$$\frac{(t - t_1)(t_2 - t)}{(t_2 - t_1)}.$$

Proof. $X(t)$ subject to the conditions $X(t_1) = A$ and $X(t_2) = B$ has the same density as the random variable $A + X(t - t_1)$, under the condition $X(t_2 - t_1) = B - A$ by condition 2 of the definition of Brownian motion. Then apply the theorem with $s = t - t_1$ and $t = t_2 - t_1$. □

Sources


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Algorithms, Scripts, Simulations

Algorithm

Simulate a sample path of the Wiener process as follows, see [1]. Divide the interval $[0, T]$ into a grid $0 = t_0 < t_1 < \ldots < t_{N-1} < t_N = T$ with $t_{i+1} - t_i = \Delta t$. Set $i = 1$ and $W(0) = W(t_0) = 0$ and iterate the following algorithm.

1. Generate a new random number $z$ from the standard normal distribution.

2. Set $i$ to $i + 1$. 
3. Set \( W(t_i) = W(t_{i-1}) + z\sqrt{\Delta t} \).

4. If \( i < N \), iterate from step 1.

This method of approximation is valid only on the points of the grid. In between any two points \( t_i \) and \( t_{i-1} \), the Wiener process is approximated by linear interpolation.

**Scripts**

**GeoGebra**

R script for Wiener process.

```r
N <- 100
# number of end-points of the grid including T
T <- 1
# length of the interval [0, T] in time units
Delta <- T/N
# time increment
W <- numeric(N + 1)
# initialization of the vector W approximating
# Wiener process

\[ t <- \text{seq}(0, T, \text{length}=N + 1) \]

\[ W <- c(0, \text{cumsum}(\sqrt{\text{Delta}} * \text{rnorm(N)})) \]

\[ \text{plot}(t, W, \text{type}="l", \text{main}="\text{Wiener process}\), \text{ylim}=c(-1,1) \)
```

Octave

Octave script for Wiener process

```octave
N = 100;
# number of end-points of the grid including T
T = 1;
# length of the interval [0, T] in time units
Delta = T/N;
# time increment
W = zeros(1, N+1);
# initialization of the vector W approximating
# Wiener process

\[ t = (0:Delta:T) \]

\[ W(2:N+1) = \text{cumsum}(\sqrt{\text{Delta}} * \text{stdnormal_rnd}(1,N)) \]

\[ \text{plot}(t, W) \]
```
use PDL::NiceSlice;

# number of end-points of the grid including T
$N = 100;

# length of the interval [0, T] in time units
$T = 1;

# time increment
$Delta = $T / $N;

# initialization of the vector W approximating
# Wiener process
$W = zeros( $N + 1 );
$W ( 1 : $N ) .= cumusumover( sqrt($Delta) * grandom($N) );

print "Simulation of the Wiener process:", $W, "\n";

## optional file output to use with external plotting
## such as gnuplot, R, octave, etc.
## Start gnuplot, then from gnuplot prompt
## plot "wienerprocess.dat" with lines
## open( F, ">wienerprocess.dat" ) || die "cannot write: $!";
# foreach $j ( 0 .. $N ) {
# print F $t->range( [$j] ), " ", $W->range( [$j] ),
# "\n";
# }
# close(F);
```python
import scipy

N = 100
T = 1.
Delta = T/N

# initialization of the vector W approximating
# Wiener process
W = scipy.zeros(N+1)

t = scipy.linspace(0, T, N+1);
W[1:N+1] = scipy.cumsum(scipy.sqrt(Delta)*scipy.random.standard_normal(N))

print "Simulation of the Wiener process:\n", W

## optional file output to use with external plotting
## such as gnuplot, R, octave, etc.
## Start gnuplot, then from gnuplot prompt
## plot "wienerprocess.dat" with lines
# f = open ('wienerprocess.dat ', 'w ')
# for j in range (0 ,N+1):
#   f.write( str(t[j])+ ' '+str(W[j])+\n ');
# f.close()
```

Problems to Work for Understanding

1. Let $W(t)$ be standard Brownian motion.
   
   (a) Find the probability that $0 < W(1) < 1$.
   
   (b) Find the probability that $0 < W(1) < 1$ and $1 < W(2) - W(1) < 3$. 

13
(c) Find the probability that $0 < W(1) < 1$ and $1 < W(2) - W(1) < 3$ and $0 < W(3) - W(2) < 1/2$.

2. Let $W(t)$ be standard Brownian motion.

(a) Find the probability that $0 < W(1) < 1$.

(b) Find the probability that $0 < W(1) < 1$ and $1 < W(2) < 3$.

(c) Find the probability that $0 < W(1) < 1$ and $1 < W(2) < 3$ and $0 < W(3) < 1/2$.

(d) Explain why this problem is different from the previous problem, and also explain how to numerically evaluate to the probabilities.

3. Explicitly write the joint probability density function for $W(t_1) = x_1$ and $W(t_2) = x_2$.

4. Let $W(t)$ be standard Brownian motion.

(a) Find the probability that $W(5) \leq 3$ given that $W(1) = 1$.

(b) Find the number $c$ such that $\Pr[W(9) > c \mid W(1) = 1] = 0.10$.

5. Let $Z$ be a normally distributed random variable, with mean 0 and variance 1, $Z \sim N(0, 1)$. Then consider the continuous time stochastic process $X(t) = \sqrt{t}Z$. Show that the distribution of $X(t)$ is normal with mean 0 and variance $t$. Is $X(t)$ a Brownian motion?

6. Let $W_1(t)$ be a Brownian motion and $W_2(t)$ be another independent Brownian motion, and $\rho$ is a constant between $-1$ and 1. Then consider the process $X(t) = \rho W_1(t) + \sqrt{1 - \rho^2}W_2(t)$. Is this $X(t)$ a Brownian motion?

7. What is the distribution of $W(s) + W(t)$, for $0 \leq s \leq t$? (Hint: Note that $W(s)$ and $W(t)$ are not independent. But you can write $W(s) + W(t)$ as a sum of independent variables. Done properly, this problem requires almost no calculation.)

8. For two random variables $X$ and $Y$, statisticians call

$$\text{Cov}(X, Y) = E[(X - E[X])(Y - E[Y])]$$
the covariance of $X$ and $Y$. If $X$ and $Y$ are independent, then $\Cov(X, Y) = 0$. A positive value of $\Cov(X, Y)$ indicates that $Y$ tends to increases as $X$ does, while a negative value indicates that $Y$ tends to decrease when $X$ increases. Thus, $\Cov(X, Y)$ is an indication of the mutual dependence of $X$ and $Y$. Show that

$$\Cov(W(s), W(t)) = E[W(s)W(t)] = \min(t, s)$$

9. Show that the probability density function

$$p(t; x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - y)^2}{2t}\right)$$

satisfies the partial differential equation for heat flow (the heat equation)

$$\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}$$

10. Change the scripts to simulate the Wiener process

(a) over intervals different from $[0, 1]$, both longer and shorter,

(b) with more grid points, that is, smaller increments,

(c) with several simulations on the same plot.

Discuss how changes in the parameters of the simulation change the Wiener process.

11. Choose a stock index such as the S & P 500, the Wilshire 5000, etc., and obtain closing values of that index for a year-long (or longer) interval of trading days. Find the variance of the closing values and create a random walk on the same interval with the same initial value and variance. Plot both sets of data on the same axes, as in Figure 1. Discuss the similarities and differences.

12. Choose an individual stock or a stock index such as the S & P 500, the Wilshire 5000, etc., and obtain values of that index at regular intervals such as daily or hourly for a long interval of trading. Find the regular differences, and normalize by subtracting the mean and dividing by the standard deviation. Simultaneously plot a histogram of the differences and the standard normal probability density function. Discuss the similarities and differences.
Reading Suggestion:

References


Outside Readings and Links:

1. [Copyright 1967 by Princeton University Press, Edward Nelson](#) On line book *Dynamical Theories of Brownian Motion*. It has a great historical review about Brownian motion.

2. [National Taiwan Normal University, Department of Physics](#) A simulation of Brownian motion that also allows you to change certain parameters.

3. [Department of Mathematics, University of Utah, Jim Carlson](#) A Java applet demonstrates Brownian Paths noticed by Robert Brown.
4. **Department of Mathematics, University of Utah, Jim Carlson** Some applets demonstrate Brownian motion, including Brownian paths and Brownian clouds.

5. **School of Statistics, University of Minnesota, Twin Cities, Charlie Geyer** An applet that draws one-dimensional Brownian motion.

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