Stochastic Processes and
Advanced Mathematical Finance

Limitations of the Black-Scholes Model

Rating

Student: contains scenes of mild algebra or calculus that may require guidance.
Section Starter Question

We have derived and solved the Black-Scholes equation. We have derived parameter dependence and sensitivity of the solution. Are we done? What’s next? How should we go about implementing and analyzing that next step, if any?

Key Concepts

1. The Black-Scholes model overprices “at the money” call options, that is with \( S \approx K \). The Black-Scholes model underprices call options at the ends, either deep “in the money”, \( S \gg K \), or deep “out of the money”, \( S \ll K \).

2. This is an indication that security price processes have “fat tails”, i.e. a distribution which has the probability of large changes in price \( S \) greater than would be predicted by the lognormal distribution.

3. Mathematical models in finance do not have the same experimental basis and long experience as do mathematical models in physical sciences. It is important to remember to apply mathematical models only under circumstances where the assumptions apply.

4. Financial economists and mathematicians have suggested alternatives to the Black-Scholes model. These alternatives include:

   - models where the future volatility of a stock price is uncertain (called stochastic volatility models); and
   - models where the stock price experiences occasional jumps rather than continuous change (called jump-diffusion models).
Vocabulary

1. The distribution of price changes for many securities exhibit **fat tails**: stock price changes far from the mean are more likely than the normal distribution predicts.

2. **Stochastic volatility** models are higher-order mathematical finance models where the volatility of a security price is a stochastic process itself.

3. **Jump-diffusion models** are higher-order mathematical finance models where the security price experiences occasional jumps rather than continuous change.

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Mathematical Ideas

Validity of Black-Scholes

Recall that the Black-Scholes Model is based on several assumptions:

1. The price of the underlying security for which we are considering a derivative financial instrument follows the stochastic differential equation

   \[ dS = rS \, dt + \sigma S \, dW \]

   or equivalently that \( S(t) \) is a Geometric Brownian Motion

   \[ S(t) = z_0 \exp((r - \sigma^2/2)t + \sigma W(t)). \]
At each time the Geometric Brownian Motion has a lognormal distribution with parameters $(\ln(z_0) + rt - \sigma^2/2t)$ and $\sigma \sqrt{t}$. The mean value of the Geometric Brownian Motion is $\mathbb{E}[S(t)] = z_0 \exp(rt)$.

2. The risk free interest rate $r$ and volatility $\sigma$ are constants.

3. The value $V$ of the derivative depends only on the current value of the underlying security $S$ and the time $t$, so we can write $V(S, t)$.

4. All variables are real-valued, and all functions are sufficiently smooth to justify necessary calculus operations.

See [Derivation of the Black-Scholes Equation](#) for the context of these assumptions.

One judgment on the validity of these assumptions statistically compares the predictions of the Black-Scholes model with the market prices of call options. This is the observation or validation phase of the cycle of mathematical modeling, see [Brief Remarks on Math Models](#) for the cycle and diagram. A detailed examination (the financial and statistical details of this examination are outside the scope of these notes) shows that the Black-Scholes formula misprices options. In fact, the Black-Scholes model overprices “at the money” options, that is with $S \approx K$, and underprices options at the ends, either deep “in the money”, $S \gg K$, or deep “out of the money”, $S \ll K$. Since the actual option price is higher than the price from the Black-Scholes formula, this indicates that the market assigns greater risk to events that are far from the center than the model would predict.

Another test of some of these assumptions is to gather data on the actual market price of call options on a security, all with the same expiration date, with a variety of strike prices. Then one can compute the implied volatility from the data. The implied volatility is not constant, in spite of our assumption of constant volatility! The computed implied volatility is higher for either deep “in the money”, $S_0 \gg K$, or deep “out of the money”, $S_0 \ll K$, than “at the money”, $S_0 \approx K$, where $S_0$ is the current asset price and $K$ is the strike price. The resulting concave up shape is called the “volatility smile” so called because it looks like a cartoon happy-face smile. The greater implied volatility far from the center indicates that the market assigns greater risk to events that are far from the center. Figure 1 shows actual data on implied volatility. The data is from a call option on the S & P 500 index price on July 1, 2013, with an expiration date of July 5, 2013, just 4 calendar...
days later. The S & P 500 price $S_0$ on July 1, 2013 was 1614.96, shown with the vertical dashed line. The implied volatility in this example is large at the ends since the S & P 500 index would have to be very volatile to change by 200 to 400 points in just 3 trading days. (July 4 is a holiday.) This example clearly illustrates that the market does not value options consistently with the constant volatility assumptions of the Black-Scholes model.

Some studies have shown that the event of downward jumps three standard deviations below the mean is three times more likely than a normal distribution would predict. This means that if we used Geometric Brownian Motion to compute the historical volatility of the S&P 500, we would find that the normal theory seriously underestimates the likelihood of large downward jumps. Jackwerth and Rubinstein [4] observe that with the Geometric Brownian Motion model, the crash of 1987 is an impossibly unlikely event:

Take for example the stock market crash of 1987. Following the standard paradigm, assume that the stock market returns are log-normally distributed with an annualized volatility of 20%. . . On October 19, 1987, the two-month S&P 500 futures price fell 29%. Under the log-normal hypothesis, this [has a probability of] $10^{-160}$. Even if one were to have lived through the 20 billion year life of the universe . . . 20 billion times . . . that such a decline could have happened even once in this period is virtually impossible.
The popular term for such extreme changes is a “black swan”, reflecting the rarity of spotting a black swan among white swans. In financial markets “black swans” occur much more often than the standard probability models predict \[10, 7\]. All these empirical tests indicate that the price process has “fat tails”, i.e. a distribution which has the probability of large changes in price \( S \) larger than the lognormal distribution predicts, \[11\] page 9]. The assumption that the underlying security has a price modeled by Geometric Brownian Motion, or equivalently that at any time the security price has a lognormal distribution is incorrect. Large changes are more frequent than the model expects.

An example of a fat-tailed distribution is the Cauchy distribution with density function \( f_C(x) = \frac{1}{\pi} \frac{1}{1+x^2} \). The rate of decay to 0 of the density as \( x \to \infty \) is the inverse power law \( x^{-2} \). This is much slower than the rate of decay of the normal distribution density function \( f_N(x) = \frac{1}{\sqrt{2\pi}} \exp(-x^2) \). For large values, the probability of events governed by the Cauchy distribution is greater than the probability of events governed by the normal distribution.

An obstacle to using the Cauchy distribution is that its variance is undefined (or more loosely stated, the variance is infinite.) The assumption of finite variance is useful for theoretical analysis of probabilities. Consider how many times in this text the assumption that the variance of a distribution was finite was used.
Another example of a fat-tailed distribution is the Student’s t-distribution with density function

\[ f_S(t) = \frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\nu\pi} \Gamma\left(\frac{\nu}{2}\right)} \left(1 + \frac{t^2}{\nu}\right)^{-\frac{\nu+1}{2}}. \]

This distribution is intermediate in the sense that for \( \nu = 1 \) it is the Cauchy distribution, while as \( \nu \to \infty \), it approaches the normal distribution. For \( \nu > 1 \) the variance of the distribution is finite, but now there is another parameter \( \nu \) that must be fit. Lions [5] finds that a Student’s t-distribution with \( \nu = 3.4 \) provides a good fit to the distribution of the daily returns of the S & P 500.

More fundamentally, one can look at whether general market prices and security price changes fit the hypothesis of following the stochastic differential equation for Geometric Brownian Motion. Studies of security market returns reveal an important fact: As in Lions [5], large changes in security prices are more likely than normally distributed random effects would predict. Put another way, the stochastic differential equation model predicts that large price changes are much less likely than is actually the case.

The origin of the difference between the stochastic differential equation model for Geometric Brownian Motion and real financial markets may be a fundamental misapplication of probability modeling. The mathematician Benoît Mandelbrot argues that finance is prone to a “wild randomness” not usually seen in traditional applications of statistics [11]. Mandelbrot says that rare big changes can be more significant than the sum of many small changes. That is, Mandelbrot calls into question the applicability of the Central Limit Theorem in finance. It may be that the mathematical hypotheses required for the application of the Central Limit Theorem do not hold in finance.

Recall that we explicitly assumed that many of the parameters were constant, in particular, volatility is assumed constant. Actually, we might wish to relax that idea somewhat, and allow volatility to change in time. Non-constant volatility introduces another dimension of uncertainty and also of variability into the problem. Still, changing volatility is an area of active research, both practically and academically.

We also assumed that trading was continuous in time, and that security prices moved continuously. Of course, continuous change is an idealizing assumption. In fact, in October 1987, the markets dropped suddenly,
almost discontinuously, and market strategies based on continuous trading were not able to keep with the selling panic. The October 1987 drop is yet another illustration that the markets do not behave exactly as the assumptions made. Another relaxation of the Black-Scholes model is assume that the price process can make sudden discontinuous jumps, leading to what are called jump-diffusion models. The mathematics associated with these processes is necessarily more complicated. This is another area of active research. [2].

**Black-Scholes as an Approximate Model with an Exact Solution**

In the classification of the section on mathematical modeling [Brief Remarks on Mathematical Models] the Black-Scholes model is an approximate model with an exact solution. The previous paragraphs show that each of the assumptions made to derive the Black-Scholes equation is an approximation made expressly to derive a tractable model. The assumption that security prices act as a Geometric Brownian Motion means that we can use the techniques of Itô stochastic calculus. Even more so, the resulting lognormal distribution is based on the normal distribution which has simple mathematical properties. The assumptions that the risk free interest rate \( r \) and volatility \( \sigma \) are constants simplifies the derivation and reduces the size of the resulting model. The assumption that the value \( V \) of the derivative depends only on the current value of the underlying security \( S \) and the time \( t \) likewise reduces the size of the resulting model. The assumption that all variables are real-valued, and all functions are sufficiently smooth allows necessary calculus operations.

Once the equation is derived we are able to solve it with standard mathematical techniques. In fact, the resulting Black-Scholes formula is expressed in a single line as a combination of well-known functions. The formula is simple enough that by 1975 Texas Instruments created a hand-held calculator specially programmed to produce Black-Scholes option prices and hedge ratios. The simple exact solution puts the Black-Scholes equation in the same position as the pendulum equation and the Ideal Gas Law, an approximate model with an exact solution. The trust we have in solutions to mathematical equations creates a sanitizing effect when we use mathematical models. The mere fact that sophisticated mathematical methods created a solution makes
a conclusion, faulty or not, seem inevitable and unimpeachable. [8, page 4]. Although an exact solution is satisfying, even beautiful, exact solutions hide the approximate origins. We often interpret the results of mathematical analysis as “objective” when it is only as objective as the underlying assumptions. The deep mathematical analysis relies in complex ways on the assumptions, so the result is an apparently strong, objective result that is actually neither strong nor objective. In the worst cases, such analysis is an example of the “garbage in, garbage out” problem, [8, page 6]. It is tempting to assume you have the perfect answer when at best you have a noisy approximation, [8, page 2].

The elegant one-line solution may actually encourage some of the misuses of mathematical modeling detailed below. Once the elegant solution is written and even programmed in a calculator, everyone can use it. Had the Black-Scholes equation been nonlinear or based on more sophisticated stochastic processes, it probably would not have a closed-form solution. Solutions would have relied on numerical calculation, more advanced mathematical techniques or simulation, that is, solutions would not have been easily obtained. Those end-users who persevered in applying the theory might have sought additional approximations more conscious of the compromises with reality.

**Misuses of Mathematical Modeling**

By 2005, about 5% of jobs in the finance industry were in mathematical finance. The heavy use of flawed mathematical models contributed to the failure and near-failure of some Wall Street firms in 2009. Some critics blamed the mathematics and the models for the general economic troubles that resulted. In spite of the flaws, mathematical modeling in finance is not going away. Consequently, modelers and users have to be honest and aware of the limitations in mathematical modeling, [11]. Mathematical models in finance do not have the same experimental basis and long experience as do mathematical models in physical sciences. For the time being, we should cautiously use mathematical models in finance as good indicators that point to the values of financial instruments, but do not predict with high precision.

Actually, the problem goes deeper than just realizing that the precise distribution of security price movements is slightly different from the assumed lognormal distribution. Even if the probability distribution type is specified, giving a mathematical description of the risk, we still would have the uncer-
tainty of not knowing the precise parameters of the distribution to specify it totally. From a scientific point of view, the way to estimate the parameters is to statistically evaluate the outcomes from the past to determine the parameters. We looked at one case of this when we described historical volatility as a way to determine $\sigma$ for the lognormal distribution, see Implied Volatility. However, this implicitly assumes that the past is a reasonable predictor of the future. While this faith is justified in the physical world where physical parameters do not change, such a faith in constancy is suspect in the human world of the markets. Consumers, producers, and investors all change habits suddenly in response to fads, bubbles, rumors, news, and real changes in the economic environment. Their change in economic behavior changes the parameters.

Even within finance, the models may vary in applicability. Analysis of the 2008-2009 market collapse indicates that the markets for interest rates and foreign exchange may have followed the models, but the markets for debt obligations may have failed to take account of low-probability extreme events such as the fall in house prices [11]. The models for debt obligations may have also assumed independence of events that were actually connected and correlated.

Models can have other problems that are more social than mathematical. Sometimes the use of the models change the market priced by the model. This feedback process in economics has been noted with the Black-Scholes model, [11]. Sometimes special financial instruments can be so complex that modeling them requires too many assumptions, yet the temptation to make an approximate model with an exact solution overtakes the solution step in the modeling process. Special debt derivatives called “collateralized debt obligations” or CDOs implicated in the economic collapse of 2008 are an example. Each CDO was a unique mix of assets, but CDO modeling used general assumptions that were not associated with the specific mix. Additionally, the CDO models used assumptions which underestimated the correlation of movements of the parts of the mix [11]. Valencia [11] says that the “The CDO fiasco was an egregious and relatively rare case of an instrument getting way ahead of the ability to map it mathematically.”

It is important to remember to apply mathematical models only under circumstances where the assumptions apply [11]. For example “Value At Risk” or VAR models use volatility to statistically estimate the likelihood that a given portfolio’s losses will exceed a certain amount. However, VAR works only for liquid securities over short periods in normal markets. VAR
cannot predict losses under sharp unexpected drops that are known to occur more frequently than expected under simple hypotheses. Mathematical economists, especially Taleb, have heavily criticized the misuse of VAR models.

**Alternatives to Black-Scholes**

Financial economists and mathematicians have suggested several alternatives to the Black-Scholes model. These alternatives include:

- **stochastic volatility** models where the future volatility of a security price is uncertain; and

- **jump-diffusion models** where the security price experiences occasional jumps rather than continuous change.

The difficulty in mathematically analyzing these models and the typical lack of closed-form solutions means that these models are not as widely known or celebrated as the Black-Scholes-Merton theory.

In spite of these flaws, the Black-Scholes model does an adequate job of generally predicting market prices. Generally, the empirical research is supportive of the Black-Scholes model. Observed differences have been small (but real!) compared with transaction costs. Even more importantly for mathematical economics, the Black-Scholes model shows how to assign prices to risky assets by using the principle of no-arbitrage applied to a replicating portfolio and reducing the pricing to applying standard mathematical tools.

**Sources**

Problems to Work for Understanding

1. A pharmaceutical company has a stock that is currently $25. Early to-
morrow morning the Food and Drug Administration will announce that it has either approved or disapproved for consumer use the company’s cure for the common cold. This announcement will either immediately increase the stock price by $10 or decrease the price by $10. Discuss the merits of using the Black-Scholes formula to value options on the stock.

2. Consider

1. A standard normal random variable $Z$ with pdf $f_Z(x) = e^{-x^2/2}/\sqrt{2\pi}$,
2. a “double exponential random variable” $X$ with pdf $f_X(x) = (1/2)e^{-|x|}$,
3. a standard uniform random variable $U$ with pdf $f_U(x) = 1$ for $x = [-1/2, 1/2]$ and 0 elsewhere, and
4. a Cauchy random variable $Y$ with pdf $f_Y(x) = 1/(\pi(1 + x^2))$.

   (a) On the same set of axes with a horizontal axis $-5 \leq x \leq 5$, graph the pdf of each of these random variables.

   (b) For each of the random variables $Z$, $X$, $U$ and $Y$ calculate the probability that the r.v. is greater than 1, 2, 3, 4. Make a well-organized and labeled table to display your results.

   (c) On the basis of the answers from the previous two questions, rank the random variables $Z$, $X$, $U$ and $Y$ in terms of the fatness of the tails.

3. Show that the variance of the Cauchy distribution is undefined.
4. Find a source of historical data on options. These sources usually list the implied volatility along with the call and put option prices. (If the implied volatility is not listed, calculate the implied volatility.) Plot the implied volatility as a function of the strike price to illustrate the volatility smile.

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**Reading Suggestion:**

**References**


Outside Readings and Links:

1. Paul Wilmott on Quantitative Finance, Chapter 16, Fat tails
2. HistoricalOptionData.com

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