Stochastic Processes and Advanced Mathematical Finance

Sensitivity, Hedging and the “Greeks”

Rating
Mathematically Mature: may contain mathematics beyond calculus with proofs.
Section Starter Question

Recall when we first considered options, see Options.

1. What did we intuitively predict would happen to the value of a call option if the underlying security value increased?

2. What did we intuitively predict would happen to the value of a call option as the time increased to the expiration date?

3. What did we intuitively predict would happen to the value of a call option if the volatility of the underlying security value increased?

Key Concepts

1. The sensitivity of the Black-Scholes formula to each of the variables and parameters is named, is fairly easily expressed, and has important consequences for hedging investments.

2. The sensitivity of the Black-Scholes formula (or any mathematical model) to its parameters is important for understanding the model and its utility.

Vocabulary

1. The Delta ($\Delta$) of a financial derivative is the rate of change of the value with respect to the value of the underlying security, in symbols

$$\Delta = \frac{\partial V}{\partial S}.$$
2. The **Gamma** ($\Gamma$) of a derivative is the sensitivity of $\Delta$ with respect to $S$, in symbols

$$\Gamma = \frac{\partial^2 V}{\partial S^2}.$$ 

3. The **Theta** ($\Theta$) of a European claim with value function $V(S,t)$ is

$$\Theta = \frac{\partial V}{\partial t}.$$ 

4. The **rho** ($\rho$) of a derivative security is the rate of change of the value of the derivative security with respect to the interest rate, in symbols

$$\rho = \frac{\partial V}{\partial r}.$$ 

5. The **Vega** ($\Lambda$) of derivative security is the rate of change of value of the derivative with respect to the volatility of the underlying asset, in symbols

$$\Lambda = \frac{\partial V}{\partial \sigma}.$$ 

6. **Hedging** is the attempt to make a portfolio value immune to small changes in the underlying asset value (or its parameters).

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**Mathematical Ideas**

To start the examination of each of the sensitivities, restate the Black-Scholes formula for the value of a European call option:

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}}$$
and then

\[ V_C(S,t) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2). \]

Note that \( d_2 = d_1 - \sigma^2\sqrt{T-t}. \)
Delta

The Delta of a European call option is the rate of change of its value with respect to the underlying security price:

\[ \Delta = \frac{\partial V_C}{\partial S} \]

\[ = \Phi(d_1) + S \Phi'(d_1) \frac{\partial d_1}{\partial S} \]

\[ - K \exp(-r(T-t)) \Phi'(d_2) \frac{\partial d_2}{\partial S} \]

\[ = \Phi(d_1) + S \frac{1}{\sqrt{2\pi}} \exp(-d_1^2/2) \frac{1}{S\sigma\sqrt{T-t}} \]

\[ - K \exp(-r(T-t)) \frac{1}{\sqrt{2\pi}} \exp(-d_2^2/2) \frac{1}{S\sigma\sqrt{T-t}} \]

\[ = \Phi(d_1) + S \frac{1}{\sqrt{2\pi}} \exp(-d_1^2/2) \frac{1}{S\sigma\sqrt{T-t}} \]

\[ - K \exp(-r(T-t)) \frac{1}{\sqrt{2\pi}} \exp \left( - \left( d_1 - \sigma\sqrt{T-t} \right)^2 / 2 \right) \frac{1}{S\sigma\sqrt{T-t}} \]

\[ = \Phi(d_1) + \exp(-d_1^2/2) \sqrt{2\pi\sigma\sqrt{T-t}} \]

\[ \left[ 1 - \frac{K \exp(-r(T-t))}{S} \exp \left( d_1 \sigma\sqrt{T-t} - \sigma^2(T-t)/2 \right) \right] \]

\[ = \Phi(d_1) + \exp(-d_1^2/2) \sqrt{2\pi\sigma\sqrt{T-t}} \]

\[ \left[ 1 - \frac{K \exp(-r(T-t))}{S} \exp \left( \log(S/K) + (r + \sigma^2/2)(T-t) - \sigma^2(T-t)/2 \right) \right] \]

\[ = \Phi(d_1) + \exp(-d_1^2/2) \sqrt{2\pi\sigma\sqrt{T-t}} \]

\[ \left[ 1 - \frac{K \exp(-r(T-t))}{S} \exp \left( \log(S/K) + r(T-t) \right) \right] \]

\[ = \Phi(d_1) \]

Figure 2 is a graph of Delta as a function of S for several values of t. Note that since 0 < \( \Phi(d_1) < 1 \) (for all reasonable values of \( d_1 \)), \( \Delta > 0 \), and so the value of a European call option is always increasing as the underlying security value increases. This is precisely as we intuitively predicted when
we first considered options, see Options. The increase in security value in $S$ is visible in Figure ??.

**Delta Hedging**

Notice that for any sufficiently differentiable function $F(S)$

$$F(S_1) - F(S_2) \approx \frac{dF}{dS} \cdot (S_1 - S_2).$$

Therefore, for the Black-Scholes formula for a European call option, using our current notation $\Delta = \frac{\partial V}{\partial S}$,

$$(V(S_1) - V(S_2)) - \Delta \cdot (S_1 - S_2) \approx 0.$$ 

Equivalently for small changes in security price from $S_1$ to $S_2$,

$$V(S_1) - \Delta \cdot S_1 \approx V(S_2) - \Delta \cdot S_2.$$ 

In financial language, we express this relationship as:

“Being long 1 derivative and short $\Delta$ units of the underlying asset is approximately market neutral for small changes in the asset value.”

We say that the sensitivity of the financial derivative value with respect to the asset value, denoted $\Delta$, gives the **hedge-ratio**. The hedge-ratio is
the number of short units of the underlying asset which combined with a call option will offset immediate market risk. After a change in the asset value, \( \Delta(S) \) will also change, and so we will need to dynamically adjust the hedge-ratio to keep pace with the changing asset value. Thus \( \Delta(S) \) as a function of \( S \) provides a dynamic strategy for hedging against risk.

We have seen this strategy before. In the derivation of Black-Scholes equation, we required that the amount of security in our portfolio, namely \( \phi(t) \), be chosen so that \( \phi(t) = V_S \). (See Derivation of the Black-Scholes Equation) The choice \( \phi(t) = V_S \) gave us a risk-free portfolio that must change in the same way as a risk-free asset.

**Gamma: The convexity factor**

The Gamma (\( \Gamma \)) of a derivative is the sensitivity of \( \Delta \) with respect to \( S \):

\[
\Gamma = \frac{\partial^2 V}{\partial S^2}.
\]

The concept of Gamma is important when the hedged portfolio cannot be adjusted continuously in time according to \( \Delta(S(t)) \). If Gamma is small then Delta changes very little with \( S \). This means the portfolio requires only infrequent adjustments in the hedge-ratio. However, if Gamma is large, then the hedge-ratio Delta is sensitive to changes in the price of the underlying security.

According to the Black-Scholes formula,

\[
\Gamma = \frac{1}{S\sqrt{2\pi}\sigma\sqrt{T-t}} \exp(-d_1^2/2)
\]

Notice that \( \Gamma > 0 \), so the call option value is always concave-up with respect to \( S \). See this in Figure ??.

**Theta: The time decay factor**

The Theta (\( \Theta \)) of a European claim with value function \( V(S,t) \) is

\[
\Theta = \frac{\partial V}{\partial t}.
\]
Note that this definition is the rate of change with respect to the real (or calendar) time, some other authors define the rate of change with respect to the time-to-expiration \( T - t \), so be careful when reading.

The Theta of a claim is sometimes refereed to as the time decay of the claim. For a European call option on a non-dividend-paying stock,

\[
\Theta = -\frac{S \cdot \sigma}{2\sqrt{T - t}} \cdot \frac{\exp(-d_1^2/2)}{\sqrt{2\pi}} - rK \exp(-r(T - t))\Phi(d_2).
\]

Note that \( \Theta \) for a European call option is negative, so the value of a European call option is decreasing as a function of time, confirming what we intuitively deduced before. See this in Figure ??.

Theta does not act like a hedging parameter as do Delta and Gamma. Although there is uncertainty about the future stock price, there is no uncertainty about the passage of time. It does not make sense to hedge against the passage of time on an option.

Note that the Black-Scholes partial differential equation can now be written as

\[
\Theta + rS\Delta + \frac{1}{2}\sigma^2S^2\Gamma = rV.
\]

Given the parameters \( r \), and \( \sigma^2 \), and any 4 of \( \Theta \), \( \Delta \), \( \Gamma \), \( S \) and \( V \) the remaining quantity is implicitly determined.

**Rho: The interest rate factor**

The rho (\( \rho \)) of a derivative security is the rate of change of the value of the derivative security with respect to the interest rate. It measures the sensitivity of the value of the derivative security to interest rates. For a European call option on a non-dividend paying stock,

\[
\rho = K(T - t) \exp(-r(T - t))\Phi(d_2)
\]

so \( \rho \) is always positive. An increase in the risk-free interest rate means a corresponding increase in the derivative value.

**Vega: The volatility factor**

The Vega (\( \Lambda \)) of a derivative security is the rate of change of value of the derivative with respect to the volatility of the underlying asset. (Some au-
thors denote Vega by variously $\lambda$, $\kappa$, and $\sigma$; referring to Vega by the corresponding Greek letter name.) For a European call option on a non-dividend-paying stock,

$$\Lambda = S\sqrt{T-t}\exp(-d_1^2/2)$$

so the Vega is always positive. An increase in the volatility will lead to a corresponding increase in the call option value. These formulas implicitly assume that the price of an option with variable volatility (which we have not derived, we explicitly assumed volatility was a constant!) is the same as the price of an option with constant volatility. To a reasonable approximation this seems to be the case, for more details and references, see [2, page 316].

**Hedging in Practice**

It would be wrong to give the impression that traders continuously balance their portfolios to maintain Delta neutrality, Gamma neutrality, Vega neutrality, and so on as would be suggested by the continuous mathematical formulas presented above. In practice, transaction costs make frequent balancing expensive. Rather than trying to eliminate all risks, an option trader usually concentrates on assessing risks and deciding whether they are acceptable. Traders tend to use Delta, Gamma, and Vega measures to quantify the different aspects of risk in their portfolios.

**Sources**

Algorithms, Scripts, Simulations

Algorithm

For given parameter values, the Black-Scholes-Merton call option “greeks” Delta and Gamma are sampled at a specified $m \times 1$ array of times and at a specified $1 \times n$ array of security prices using vectorization and broadcasting. The result can be plotted as functions of the security price as done in the text. The calculation is vectorized for an array of $S$ values and an array of $t$ values, but it is not vectorized for arrays in the parameters $K$, $r$, $T$, and $\sigma$. This approach is taken to illustrate the use of vectorization and broadcasting for efficient evaluation of an array of solution values from a complicated formula.

In particular, the calculation of $d_1$ uses broadcasting, also called binary singleton expansion, recycling, single-instruction multiple data, threading, or replication.

The calculation relies on using the rules for calculation and handling of infinity and NaN (Not a Number) which come from divisions by 0, taking logarithms of 0, and negative numbers and calculating the normal cdf at infinity and negative infinity. The plotting routines will not plot a NaN which accounts for the gaps or omissions.

The scripts plot Delta and Gamma as functions of $S$ for the specified $m \times 1$ array of times in two side-by-side subplots in a single plotting frame.

Scripts

Geogebra GeoGebra applet

R  R script for Black-Scholes call option greeks Delta and Gamma.

```r
m <- 6
n <- 61
S0 <- 70
S1 <- 130
K <- 100
r <- 0.12
T <- 1
sigma <- 0.1

time <- seq(T, 0, length = m)
S <- seq(S0, S1, length = n)
```
numerd1 <- outer(((r + sigma^2/2) * (T - time)), log(S/K), "+")
d1 <- numerd1/(sigma * sqrt(T - time))
Delta <- pnorm(d1)

factor1 <- 1/(sqrt(2 * pi) * sigma * outer(sqrt(T - time), S, "*"))
factor2 <- exp(-d1^2/2)
Gamma <- factor1 * factor2

old.par <- par(mfrow = c(1, 2))
matplot(S, t(Delta), type = "l")
matplot(S, t(Gamma), type = "l")
par(old.par)

**Octave**  Octave script for Black-Scholes call option greeks Delta and Gamma.

m = 6;
n = 61;
S0 = 70;
S1 = 130;
K = 100;
r = 0.12;
T = 1.0;
sigma = 0.10;
time = transpose(linspace(T, 0, m));
S = linspace(S0,S1,n);
d1 = (log(S/K) + (r + sigma^2/2)*(T-time))./(sigma*sqrt(T-time));
Delta = normcdf(d1);

factor1 = 1./bsxfun(@times, sqrt(2*pi)*sigma*sqrt(T-time), S);
factor2 = exp(-d1.^2/2);
Gamma = factor1 .* factor2;

subplot(1,2,1)
plot(S,Delta)
title("Delta")
subplot(1,2,2)
plot(S,Gamma)
title("Gamma")
Perl script for Black-Scholes call option greeks Delta and Gamma.

```perl
use PDL::NiceSlice;
use PDL::Constants qw(PI);

sub pnorm {
    my ( $x , $sigma , $mu ) = @_; 
    $sigma = 1 unless defined($sigma);
    $mu = 0 unless defined($mu);
    return 0.5 * ( 1 + erf( ( $x - $mu ) / ( sqrt(2) * $sigma ) ) );
}

$m = 6;
$n = 61;
$S0 = 70;
$S1 = 130;
$K = 100;
$r = 0.12;
$T = 1.0;
$sigma = 0.10;

$time = zeroes($m)->xlinvals($T,0.0);
$S = zeroes($n)->xlinvals($S0,$S1);

$logSoverK = log ($S/$K);
$n12 = (( $r + $sigma**2/2) *($T-$time));
$numerd1 = $logSoverK + $n12->transpose;
$d1 = $numerd1/( ( $sigma * sqrt($T-$time)) -> transpose);
$Delta = pnorm($d1);

$denom1 = (sqrt(2*PI)*$sigma*sqrt($T-$time))->transpose;
$factor1 = 1/($denom1*$S);
$factor2 = exp(-$d1**2/2);
$Gamma = $factor1 * $factor2;

# file output to use with external plotting programming
# such as gnuplot, R, octave, etc.
# Start gnuplot, then from gnuplot prompt
# set key off # avoid cluttering plots
# set multiplot layout 1,2
```
SciPy  | Python script for Black-Scholes call option greeks Delta and Gamma.

```python
import scipy

m = 6
n = 61
S0 = 70
S1 = 130
K = 100
r = 0.12
T = 1.0
sigma = 0.10

time = scipy.linspace(T, 0.0, m)
S = scipy.linspace(S0, S1, n)

logSoverK = scipy.log(S / K)
n12 = (r + sigma ** 2 / 2) * (T - time)
numer1 = logSoverK[:, scipy.newaxis] + n12[:, scipy.newaxis]
d1 = numer1 / (sigma * scipy.sqrt(T - time)[:, scipy.newaxis])

from scipy.stats import norm
Delta = norm.cdf(d1)

denom1 = (scipy.sqrt(2 * scipy.pi) * sigma * scipy.sqrt(T - time))[:, scipy.newaxis]
factor1 = 1 / (denom1 * S)
factor2 = scipy.exp(-d1 ** 2 / 2)
Gamma = factor1 * factor2
```
Problems to Work for Understanding

1. How can a short position in 1,000 call options be made Delta neutral when the Delta of each option is 0.7?

2. Calculate the Delta of an at-the-money 6-month European call option on a non-dividend paying stock, when the risk-free interest rate is 10% per year (compounded continuously) and the stock price volatility is 25% per year.

3. Use the put-call parity relationship to derive the relationship between
   
   (a) the Delta of a European call option and the Delta of a European put option,
(b) the Gamma of a European call option and the Gamma of a European put option,
(c) the Vega of a European call option and a European put option, and
(d) the Theta of a European call option and a European put option.

4. (a) Derive the expression for \( \Gamma \) for a European call option.
(b) For a particular scripting language of your choice, modify the script to draw a graph of \( \Gamma \) versus \( S \) for \( K = 50, r = 0.10, \sigma = 0.25, T - t = 0.25 \).
(c) For a particular scripting language of your choice, modify the script to draw a graph of \( \Gamma \) versus \( t \) for a call option on an at-the-money stock, with \( K = 50, r = 0.10, \sigma = 0.25, T - t = 0.25 \).
(d) For a particular scripting language of your choice, modify the script to draw the graph of \( \Gamma \) versus \( S \) and \( t \) for a European call option with \( K = 50, r = 0.10, \sigma = 0.25, T - t = 0.25 \).
(e) Comparing the graph of \( \Gamma \) versus \( S \) and \( t \) with the graph of \( V_C \) versus \( S \) and \( t \) in of Solution the Black Scholes Equation, explain the shape and values of the \( \Gamma \) graph. This only requires an understanding of calculus, not financial concepts.

5. (a) Derive the expression for \( \Theta \) for a European call option, as given in the notes.
(b) For a particular scripting language of your choice, modify the script to draw a graph of \( \Theta \) versus \( S \) for \( K = 50, r = 0.10, \sigma = 0.25, T - t = 0.25 \).
(c) For a particular scripting language of your choice, modify the script to draw a graph of \( \Theta \) versus \( t \) for an at-the-money stock, with \( K = 50, r = 0.10, \sigma = 0.25, T = 0.25 \).

6. (a) Derive the expression for \( \rho \) for a European call option as given in this section.
(b) For a particular scripting language of your choice, modify the script to draw a graph of \( \rho \) versus \( S \) for \( K = 50, r = 0.10, \sigma = 0.25, T - t = 0.25 \).
7. (a) Derive the expression for Λ for a European call option as given in this section.

(b) For a particular scripting language of your choice, modify the script to draw a graph of Λ versus $S$ for $K = 50$, $r = 0.10$, $\sigma = 0.25$, $T - t = 0.25$.

8. For a particular scripting language of your choice, modify the script to create a function within that language that will evaluate the call option greeks Delta and Gamma at a time and security value for given parameters.

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**Reading Suggestion:**

**References**


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**Outside Readings and Links:**

1. [Stock Option Greeks](#) video on the meaning and interpretation of the rates of change of stock options with respect to parameters.
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