Stochastic Processes and
Advanced Mathematical Finance

Randomness

Rating

Student: contains scenes of mild algebra or calculus that may require guidance.

1
Section Starter Question

What do we mean when we say something is “random”? What is the dictionary definition of “random”?

Key Concepts

1. Assigning probability $\frac{1}{2}$ to the event that a coin will land heads and probability $\frac{1}{2}$ to the event that a coin will land tails is a mathematical model that summarizes our experience with many coins.

2. A coin flip is a deterministic physical process, subject to the physical laws of motion. Extremely narrow bands of initial conditions determine the outcome of heads or tails. The assignment of probabilities $\frac{1}{2}$ to heads and tails is a summary measure of all initial conditions that determine the outcome precisely.

3. The random walk theory of asset prices claims that market prices follow a random path without any influence by past price movements. This theory says it is impossible to predict which direction the market will move at any point, especially in the short term. More refined versions of the random walk theory postulate a probability distribution for the market price movements. In this way, the random walk theory mimics the mathematical model of a coin flip, substituting a probability distribution of outcomes for the ability to predict what will really happen.
Vocabulary

1. Technical analysis claims to predict security prices by relying on the assumption that market data, such as price, volume, and patterns of past behavior can help predict future (usually short-term) market trends.

2. The random walk theory of the market claims that market prices follow a random path up and down according to some probability distribution without any influence by past price movements. This assumption means that it is not possible to predict which direction the market will move at any point, although the probability of movement in a given direction can be calculated.

Mathematical Ideas

Coin Flips and Randomness

The simplest, most common, and in some ways most basic example of a random process is a coin flip. We flip a coin, and it lands one side up. We assign the probability $1/2$ to the event that the coin will land heads and probability $1/2$ to the event that the coin will land tails. But what does that assignment of probabilities really express?

To assign the probability $1/2$ to the event that the coin will land heads and probability $1/2$ to the event that the coin will land tails is a mathematical model that summarizes our experience with coins. We have flipped many coins many times, and we see that about half the time the coin comes up heads, and about half the time the coin comes up tails. So we abstract this observation to a mathematical model containing only one parameter, the probability of a heads.

From this simple model of the outcome of a coin flip we can derive some mathematical consequences. We will do this extensively in the chapter on
limit theorems for coin-flipping. One of the first consequences we can derive is a theorem called the Weak Law of Large Numbers. This consequence reassures us that if we make the probability assignment then long term observations with the model will match our expectations. The mathematical model shows its worth by making definite predictions of future outcomes. We will prove other more sophisticated theorems, some with reasonable consequences, others are surprising. Observations show the predictions generally match experience with real coins, and so this simple mathematical model has value in explaining and predicting coin flip behavior. In this way, the simple mathematical model is satisfactory.

In other ways the probability approach is unsatisfactory. A coin flip is a physical process, subject to the physical laws of motion. The renowned applied mathematician J. B. Keller investigated coin flips in this way. He assumed a circular coin with negligible thickness flipped from a given height \( y_0 = a > 0 \), and considered its motion both in the vertical direction under the influence of gravity, and its rotational motion imparted by the flip until the coin lands on the surface \( y = 0 \). The initial conditions imparted to the coin flip are the initial upward velocity and the initial rotational velocity. With additional simplifying assumptions Keller shows that the fraction of flips which land heads approaches \( 1/2 \) if the initial vertical and rotational velocities are high enough. Keller shows more, that for high initial velocities narrow bands of initial conditions determine the outcome of heads or tails. From Keller’s analysis we see the randomness comes from the choice of initial conditions. Because of the narrowness of the bands of initial conditions, slight variations of initial upward velocity and rotational velocity lead to different outcomes. The assignment of probabilities \( 1/2 \) to heads and tails is actually a statement of the measure of the initial conditions that determine the outcome precisely.

The assignment of probabilities \( 1/2 \) to heads and tails is actually a statement of our inability to measure the initial conditions and the dynamics precisely. The heads or tails outcomes alternate in adjacent narrow initial conditions regions, so we cannot accurately predict individual outcomes. We instead measure the whole proportion of initial conditions leading to each outcome.

If the coin lands on a hard surface and bounces, the physical prediction of outcomes is now almost impossible because we know even less about the dynamics of the bounce, let alone the new initial conditions imparted by the bounce.
Another mathematician who often collaborated with J. B. Keller, Persi Diaconis, has exploited this determinism. Diaconis, an accomplished magician, is reportedly able to flip many heads in a row using his manual skill. Moreover, he has worked with mechanical engineers to build a precise coin-flipping machine that can flip many heads in a row by controlling the initial conditions precisely. Figure 2 is a picture of such a machine.

Mathematicians Diaconis, Susan Holmes and Richard Montgomery have done an even more detailed analysis of the physics of coin flips. The coin-flipping machines help to show that flipping physical coins is actually slightly biased. Coins have a slight physical bias favoring the coin’s initial position 51% of the time. The bias results from the rotation of the coin around three axes of rotation at once. Their more complete dynamical description of coin flipping needs even more initial information.

If the coin bounces or rolls the physics becomes more complicated. This is particularly true if the coin rolls on one edge upon landing. The edges of coins are often milled with a slight taper, so the coin is really more conical than cylindrical. When landing on edge or spinning, the coin will tip in the tapered direction.

The assignment of a reasonable probability to a coin toss both summa-
izes and hides our inability to measure the initial conditions precisely and to compute the physical dynamics easily. James Gleick summarizes this neatly [2] “In physics – or wherever natural processes seem unpredictable – apparent randomness may ... arise from deeply complex dynamics.” The probability assignment is usually a good enough model, even if wrong. Except in circumstances of extreme experimental care with millions of measurements, using 1/2 for the proportion of heads is sensible.

Randomness and the Markets

A branch of financial analysis, generally called technical analysis, claims to predict security prices with the assumption that market data, such as price, volume, and patterns of past behavior predict future (usually short-term) market trends. Technical analysis also usually assumes that market psychology influences trading in a way that enables predicting when a stock will rise or fall.

In contrast is random walk theory. This theory claims that market prices follow a random path without influence by past price movements. The randomness makes it impossible to predict which direction the market will
move at any point, especially in the short term. More refined versions of the random walk theory postulate a probability distribution for the market price movements. In this way, the random walk theory mimics the mathematical model of a coin flip, substituting a probability distribution of outcomes for the ability to predict what will really happen.

If a coin flip, although deterministic and ultimately simple in execution cannot be practically predicted with well-understood physical principles, then it should be even harder to believe that some technical forecasters predict market dynamics. Market dynamics depend on the interactions of thousands of variables and the actions of millions of people. The economic principles at work on the variables are incompletely understood compared with physical principles. Much less understood are the psychological principles that motivate people to buy or sell at a specific price and time. Even allowing that economic principles which might be mathematically expressed as unambiguously as the Lagrangian dynamics of the coin flip determine market prices, that still leaves the precise determination of the initial conditions and the parameters.

It is more practical to admit our inability to predict using basic principles and to instead use a probability distribution to describe what we see. In this text, we use the random walk theory with minor modifications and qualifications. We will see that random walk theory leads to predictions we can test against evidence, just as a coin-flip sequence can be tested against the classic limit theorems of probability. In certain cases, with extreme care, special tools and many measurements of data we may be able to discern biases, even predictability in markets. This does not invalidate the utility of the less precise first-order models that we build and investigate. All models are wrong, but some models are useful.

The cosmologist Stephen Hawking says in his book *A Brief History of Time* [3] “A theory is a good theory if it satisfies two requirements: it must accurately describe a large class of observations on the basis of a model that contains only a few arbitrary elements, and it must make definite predictions about the results of future observations.” As we will see the random walk theory of markets does both. Unfortunately, technical analysis typically does not describe a large class of observations and usually has many arbitrary elements.
**True Randomness**

The outcome of a coin flip is physically determined. The numbers generated by an “random-number-generator” algorithm are deterministic, and are more properly known as pseudo-random numbers. The movements of prices in a market are governed by the hopes and fears of presumably rational human beings, and so might in principle be predicted. For each of these, we substitute a probability distribution of outcomes as a sufficient summary of what we have experienced in the past but are unable to predict precisely. Does true randomness exist anywhere? Yes, in two deeper theories, *algorithmic complexity theory* and *quantum mechanics*.

In algorithmic complexity theory, a number is *not* random if it is computable, that is, if a computer program will generate it, [2]. Roughly, a computable number has an algorithm that will generate its decimal digit expression. For example, for a rational number the division of the denominator into the numerator determines the repeating digit blocks of the decimal expression. Therefore rational numbers are not random, as one would expect. Irrational square roots are not random since a simple algorithm determines the digits of the nonterminating, nonrepeating decimal expression. Even the mathematical constant $\pi$ is not random since a short formula can generate the digits of $\pi$.

In the 1960s mathematicians A. Kolmogorov and G. Chaitin were looking for a true mathematical definition of randomness. They found one in the theory of information: they noted that if a mathematician could produce a sequence of numbers with a computer program significantly shorter than the sequence, then the mathematician would know the digits were not random. In the algorithm, the mathematician has a simple theory that accounts for a large set of facts and allows for prediction of digits still to come, [2]. Remarkably, Kolmogorov and Chaitin showed that many real numbers do not fit this definition and therefore are random. One way to describe such non-computable or random numbers is that they are not predictable, containing nothing but one surprise after another.

This definition helps explain a paradox in probability theory. Suppose we roll a fair die 20 times. One possible result is 11111111111111111111 and another possible result is 6623441536125563152. Which result is more probable to occur? Each sequence of numbers is equally likely to occur, with probability $1/6^{20}$. However, our intuition of algorithmic complexity tells us the short program “repeat 1 20 times” gives 1111111111111111, so it seems to
be not random. A description of 66234441536125563152 requires 20 separate specifications, just as long as the number sequence itself. We then believe the first monotonous sequence is not random, while the second unpredictable sequence is random. Neither sequence is long enough to properly apply the theory of algorithmic complexity, so the intuition remains vague. The paradox results from an inappropriate application of a definition of randomness. Furthermore, the second sequence has $20!/(3!^3 \cdot 4!) = 3,259,095,840,000$ permutations but there is only one permutation of the first. Instead of thinking of the precise sequence we may confuse it with the more than $3 \times 10^{12}$ other permutations and believe it is therefore more likely. The confusion of the precise sequence with the set of permutations contributes to the paradox.

In the quantum world the time until the radioactive disintegration of a specific N-13 atom to a C-13 isotope is apparently truly random. It seems we fundamentally cannot determine when it will occur by calculating some physical process underlying the disintegration. Scientists must use probability theory to describe the physical processes associated with true quantum randomness.

Einstein found this quantum theory hard to accept. His famous remark is that “God does not play at dice with the universe.” Nevertheless, experiments have confirmed the true randomness of quantum processes. Some results combining quantum theory and cosmology imply even more profound and bizarre results. Again in the words of Stephen Hawking, “God not only plays dice. He also sometimes throws the dice where they cannot be seen.”

Sources

Problems to Work for Understanding

Reading Suggestion:

References


Outside Readings and Links:

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