Stochastic Processes and
Advanced Mathematical Finance

Options and Derivatives

Rating
Student: contains scenes of mild algebra or calculus that may require guidance.
Section Starter Question

Suppose your rich neighbor offered an agreement to you today to sell his classic Jaguar sports-car to you (and only you) a year from today at a reasonable price agreed upon today. (Cash and car would be exchanged a year from today.) What would be the advantages and disadvantages to you of such an agreement? Would that agreement be valuable? How would you determine how valuable that agreement is?

Key Concepts

1. A call option is the right to buy an asset at an established price at a certain time.

2. A put option is the right to sell an asset at an established price at a certain time.

3. A European option may only be exercised at the end of its life on the expiry date, an American option may be exercised at any time during its life up to the expiry date.

4. Six factors affect the price of a stock option:
   - the current stock price $S$;
   - the strike price $K$;
   - the time to expiration $T - t$ where $T$ is the expiration time and $t$ is the current time;
   - the volatility of the stock price $\sigma$;
   - the risk-free interest rate $r$; and
   - the dividends expected during the life of the option.
Vocabulary

1. A call option is the right to buy an asset at an established price at a certain time.
2. A put option is the right to sell an asset at an established price at a certain time.
3. A future is a contract to buy (or sell) an asset at an established price at a certain time.
4. Volatility is a measure of the variability and therefore the risk of a price, usually the price of a security.

Mathematical Ideas

Definitions

A call option is the right to buy an asset at an established price at a certain time. A put option is the right to sell an asset at an established price at a certain time. Another slightly simpler financial instrument is a future which is a contract to buy or sell an asset at an established price at a certain time.

More fully, a call option is an agreement or contract by which at a definite time in the future, known as the expiry date, the holder of the option may purchase from the option writer an asset known as the underlying asset for a definite amount known as the exercise price or strike price.
A put option is an agreement or contract by which at a definite time in the future, known as the expiry date, the holder of the option may sell to the option writer an asset known as the underlying asset for a definite amount known as the exercise price or strike price. A European option may only be exercised at the end of its life on the expiry date. An American option may be exercised at any time during its life up to the expiry date. For comparison, in a futures contract the writer must buy (or sell) the asset to the holder at the agreed price at the prescribed time. The underlying assets commonly traded on options exchanges include stocks, foreign currencies, and stock indices. For futures, in addition to these kinds of assets the common assets are commodities such as minerals and agricultural products. In this text we will usually refer to options based on stocks, since stock options are easily described, commonly traded and prices are easily found.

Jarrow and Protter [2, page 7] tell a story about the origin of the names European options and American options. While writing his important 1965 article on modeling stock price movements as a geometric Brownian motion, Paul Samuelson went to Wall Street to discuss options with financial professionals. Samuelson’s Wall Street contact informed him that there were two kinds of options, one more complex that could be exercised at any time, the other more simple that could be exercised only at the maturity date. The contact said that only the more sophisticated European mind (as opposed to the American mind) could understand the former more complex option. In response, when Samuelson wrote his paper, he used these prefixes and reversed the ordering! Now in a further play on words, financial markets offer many more kinds of options with geographic labels but no relation to that place name. For example; two common types are Asian options and Bermuda options.

The Markets for Options

In the United States, some exchanges trading options are the Chicago Board Options Exchange (CBOE), the American Stock Exchange (AMEX), and the New York Stock Exchange (NYSE) among others. Not all options are traded on exchanges. Over-the-counter options markets where financial institutions and corporations trade directly with each other are increasingly popular. Trading is particularly active in options on foreign exchange and interest rates. The main advantage of an over-the-counter option is that it can be
tailored by a financial institution to meet the needs of a particular client. For example, the strike price and maturity do not have to correspond to the set standards of the exchanges. Other nonstandard features can be incorporated into the design of the option. A disadvantage of over-the-counter options is that the terms of the contract need not be open to inspection by others and the contract may be so different from standard derivatives that it is hard to evaluate in terms of risk and value.

A European put option allows the holder to sell the asset on a certain date for a prescribed amount. The put option writer is obligated to buy the asset from the option holder. If the underlying asset price goes below the strike price, the holder makes a profit because the holder can buy the asset at the current low price and sell it at the agreed higher price instead of the current price. If the underlying asset price goes above the strike price, the holder exercises the right not to sell. The put option has payoff properties that are the opposite to those of a call. The holder of a call option wants the asset price to rise, the higher the asset price, the higher the immediate profit. The holder of a put option wants the asset price to fall as low as possible. The further below the strike price, the more valuable is the put option.

The expiry date is specified by the month in which the expiration occurs. The precise expiration date of exchange traded options is 10:59 PM Central Time on the Saturday immediately following the third Friday of the expiration month. The last day on which options trade is the third Friday
of the expiration month. Exchange traded options are typically offered with lifetimes of 1, 2, 3, and 6 months.

Another item used to describe an option is the **strike price**, the price at which the asset can be bought or sold. For exchange traded options on stocks, the exchange typically chooses strike prices spaced $2.50, $5, or $10 apart. The usual rule followed by exchanges is to use a $2.50 spacing if the stock price is below $25, $5 spacing when it is between $25 and $200, and $10 spacing when it is above $200. For example, if Corporation XYZ has a current stock price of $12.25, options traded on it may have strike prices of $10, $12.50, $15, $17.50 and $20. A stock trading at $99.88 may have options traded at the strike prices of $90, $95, $100, $105, $110 and $115.

Options are called **in the money, at the money** or **out of the money**. An in-the-money option would lead to a positive cash flow to the holder if it were exercised immediately. Similarly, an at-the-money option would lead to zero cash flow if exercised immediately, and an out-of-the-money would lead to negative cash flow if it were exercised immediately. If $S$ is the stock price and $K$ is the strike price, a call option is in the money when $S > K$, at the money when $S = K$ and out of the money when $S < K$. Clearly, an option will be exercised only when it is in the money.

**Characteristics of Options**

The **intrinsic value** of an option is the maximum of zero and the value it would have if exercised immediately. For a call option, the intrinsic value is therefore $\max(S - K, 0)$. Often it might be optimal for the holder of an American option to wait rather than exercise immediately. The option is then said to have **time value**. Note that the intrinsic value does not consider the transaction costs or fees associated with buying or selling an asset.

The word “may” in the description of options, and the name “option” itself implies that for the holder of the option or contract, the contract is a **right**, and not an obligation. The other party of the contract, known as the **writer** does have a potential obligation, since the writer must sell (or buy) the asset if the holder chooses to buy (or sell) it. Since the writer confers on the holder a right with no obligation an option has some value. This right must be paid for at the time of opening the contract. Conversely, the writer of the option must be compensated for the obligation he has assumed. Our main goal is to answer the following questions:
How much should one pay for that right? That is, what is the value of an option? How does that value vary in time? How does that value depend on the underlying asset?

Note that the value of the option contract depends essentially on the characteristics of the underlying commodity. If the commodity has relatively large variations in price, then we might believe that the option contract would be relatively high-priced since with some probability the option will be in the money. The option contract value is \textit{derived} from the commodity price, and so we call it a \textit{derivative}.

Six factors affect the price of a stock option:

- the current stock price \( S \);
- the strike price \( K \);
- the time to expiration \( T - t \) where \( T \) is the expiration time and \( t \) is the current time;
- the volatility of the stock price;
- the risk-free interest rate; and
- the dividends expected during the life of the option.

Consider what happens to option prices when one of these factors changes while all the others remain fixed. The results are summarized in Table [I]. The changes regarding the stock price, the strike price, the time to expiration
Increase in Variable | European Call | European Put | American Call | American Put
--- | --- | --- | --- | ---
Stock Price | + | − | + | −
Strike Price | − | + | − | +
Time to Expiration | ? | ? | + | +
Volatility | + | + | + | +
Risk-free Rate | + | − | + | −
Dividends | − | + | − | +

Table 1: Effect of increases in the variables influencing an option price.

and the volatility are easy to explain; the other variables are less important for our considerations.

If it is to be exercised at some time in the future, the payoff from a call option will be the amount by which the stock price exceeds the strike price. Call options therefore become more valuable as the stock price increases and less valuable as the strike price increases. For a put option, the payoff on exercise is the amount by which the strike price exceeds the stock price. Put options therefore behave in the opposite way to call options. Put options become less valuable as stock price increases and more valuable as strike price increases.

Consider next the effect of the expiration date. Both put and call American options become more valuable as the time to expiration increases. The owner of a long-life option has all the exercise options open to the short-life option — and more. The long-life option must therefore, be worth at least as much as the short-life option. European put and call options do not necessarily become more valuable as the time to expiration increases. The owner of a long-life European option can only exercise at the maturity of the option.

Roughly speaking, the volatility of a stock price is a measure of how much future stock price movements may vary relative to the current price. As volatility increases, the chance that the stock price will either increase or decrease greatly relative to the present price also increases. For the owner of a stock, these two outcomes tend to offset each other. However, this is not so for the owner of a put or call option. The owner of a call benefits from price increases, but has limited downside risk in the event of price decrease since the most that he or she can lose is the price of the option. Similarly, the owner of a put benefits from price decreases but has limited upside risk.
in the event of price increases. The values of puts and calls therefore increase as volatility increases.

The reader will observe that the language about option prices in this section has been qualitative and imprecise:

- an option is “a contract to buy or sell an asset at an established price” without specifying how the price is obtained;
- “…the option contract would be relatively high-priced…”;
- “Call options therefore become more valuable as the stock price increases…” without specifying the rate of change; and
- “As volatility increases, the chance that the stock price will either increase or decrease greatly … increases”.

The goal in following sections is to develop a mathematical model which gives quantitative and precise statements about options prices and to judge the validity and reliability of the model.

Sources


Problems to Work for Understanding

1. (a) Find and write the definition of a “future”, also called a futures contract. Graph the intrinsic value of a futures contract at its contract date, or expiration date, as was done for the call option.
(b) Show that holding a call option and writing a put option on the same asset, with the same strike price \( K \) is the same as having a futures contract on the asset with strike price \( K \). Drawing a graph of the value of the combination and the value of the futures contract together with an explanation will demonstrate the equivalence.

2. Puts and calls are not the only option contracts available, just the most fundamental and the simplest. Puts and calls are designed to eliminate risk of up or down price movements in the underlying asset. Some other option contracts designed to eliminate other risks are created as combinations of puts and calls.

(a) Draw the graph of the value of the option contract composed of holding a put option with strike price \( K_1 \) and holding a call option with strike price \( K_2 \) where \( K_1 < K_2 \). (Assume both the put and the call have the same expiration date.) The investor profits only if the underlier moves dramatically in either direction. This is known as a long strangle.

(b) Draw the graph of the value of an option contract composed of holding a put option with strike price \( K \) and holding a call option with the same strike price \( K \). (Assume both the put and the call have the same expiration date.) This is called an long straddle, and also called a bull straddle.

(c) Draw the graph of the value of an option contract composed of holding one call option with strike price \( K_1 \) and the simultaneous writing of a call option with strike price \( K_2 \) with \( K_1 < K_2 \). (Assume both the options have the same expiration date.) This is known as a bull call spread.

(d) Draw the graph of the value of an option contract created by simultaneously holding one call option with strike price \( K_1 \), holding another call option with strike price \( K_2 \) where \( K_1 < K_2 \), and writing two call options at strike price \( (K_1 + K_2)/2 \). This is known as a butterfly spread.

(e) Draw the graph of the value of an option contract created by holding one put option with strike price \( K \) and holding two call options on the same underlying security, strike price, and maturity date. This is known as a triple option or strap.
Reading Suggestion:

References


Outside Readings and Links:

1. [What are stock options?](youtube.com) An explanation from youtube.com

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