Stochastic Processes and Advanced Mathematical Finance

Brief History of Mathematical Finance

Rating

Everyone.
Section Starter Question

Name as many financial instruments as you can, and name or describe the market where you would buy them. Also describe the instrument as high risk or low risk.

Key Concepts

1. **Finance theory** is the study of economic agents’ behavior allocating financial resources and risks across alternative financial instruments over time in an uncertain environment. Mathematics provides tools to model and analyze that behavior in allocation and time, taking into account uncertainty.

2. Louis Bachelier’s 1900 math dissertation on the theory of speculation in the Paris markets marks the twin births of both the continuous time mathematics of stochastic processes and the continuous time economics of option pricing.

3. The most important theoretical development was the Black-Scholes model for option pricing published in 1973.

4. The growth in sophisticated mathematical models and their adoption into financial practice accelerated during the 1980s in parallel with the extraordinary growth in financial innovation. Major developments in computing power, including the personal computer and increases in computer speed and memory enabled new financial markets and expansions in the size of existing ones.
Vocabulary

1. **Finance theory** is the study of economic agents’ behavior allocating financial resources and risks across alternative financial instruments over time in an uncertain environment.

2. A **derivative** is a financial agreement between two parties that depends on the future price or performance of an underlying asset. The underlying asset could be a stock, a bond, a currency, or a commodity.

3. **Types of derivatives**: Derivatives come in many types. The most common examples are **futures**, agreements to trade something at a set price at a given date; **options**, the right but not the obligation to buy or sell at a given price; **forwards**, like futures but traded directly between two parties instead of on exchanges; and **swaps**, exchanging one lot of obligations for another. Derivatives can be based on pretty much anything as long as two parties are willing to trade risks and can agree on a price.

Mathematical Ideas

Introduction

One sometime hears that “compound interest is the eighth wonder of the world”, or the “stock market is just a big casino”. These are colorful sayings, maybe based in happy or bitter experience, but each focuses on only one aspect of one financial instrument. The time value of money and uncertainty are the central elements influencing the value of financial instruments. Considering only the time aspect of finance, the tools of calculus and differential equations are adequate. When considering only the uncertainty, the tools of probability theory illuminate the possible outcomes. Considering time and uncertainty together, we begin the study of advanced mathematical finance.
Finance theory is the study of economic agents’ behavior allocating financial resources and risks across alternative financial instruments over time in an uncertain environment. Familiar examples of financial instruments are bank accounts, loans, stocks, government bonds and corporate bonds. Many less familiar examples abound. Economic agents are units who buy and sell financial resources in a market. Typical economic agents are individual investors, banks, businesses, mutual funds and hedge funds. Each agent has many choices of where to buy, sell, invest and consume assets. Each choice comes with advantages and disadvantages. An agent distributes resources among the many possible investments with a goal in mind, often maximum return or minimum risk.

Advanced mathematical finance is the study of the more sophisticated financial instruments called derivatives. A derivative is a financial agreement between two parties that depends on the future price or performance of an underlying asset. Derivatives are so called not because they involve a rate of change, but because their value is derived from the underlying asset. The underlying asset could be a stock, a bond, a currency, or a commodity. Derivatives have become one of the financial world’s most important risk-management tools. Finance is about shifting and distributing risk and derivatives are especially efficient for that purpose [10].

Two common derivatives are futures and options. Futures trading, a key practice in modern finance, probably originated in seventeenth century Japan, but the idea goes as far back as ancient Greece. Options were a feature of the “tulip mania” in seventeenth century Holland. Modern derivatives differ from their predecessors in that they are usually specifically designed to objectify and price financial risk.

Derivatives come in many types. The most common examples are futures, agreements to trade something at a set price at a given date; options, the right but not the obligation to buy or sell at a given price; forwards, like futures but traded directly between two parties instead of on exchanges; and swaps, exchanging flows of income from different investments to manage different risk exposure. For example, one party in a deal may want the potential of rising income from a loan with a floating interest rate, while the other might prefer the predictable payments ensured by a fixed interest rate. The name of this elementary swap is a “plain vanilla swap”. More complex swaps mix the performance of multiple income streams with varieties of risk [10]. Another more complex swap is a credit-default swap in which a seller receives a regular fee from the buyer in exchange for agreeing to cover losses
arising from defaults on the underlying loans. These swaps are somewhat like insurance [10]. These more complex swaps are the source of controversy since many people believe that they are responsible for the collapse or near-collapse of several large financial firms in late 2008. As long as two parties are willing to trade risks and can agree on a price they can craft a corresponding derivative from any financial instrument. Businesses use derivatives to shift risks to other firms, chiefly banks. About 95% of the world’s 500 biggest companies use derivatives. Markets called exchanges are the usual place to buy and sell derivatives with standardized terms. Derivatives tailored for specific purposes or risks are bought and sold “over the counter” from big banks. The “over the counter” market dwarfs the exchange trading. In November 2009, the Bank for International Settlements put the face value of over the counter derivatives at $604.6 trillion. Using face value is misleading, after stripping out off-setting claims the residual value is $3.7 trillion, still a large figure [13].

Mathematical models in modern finance contain beautiful applications of differential equations and probability theory. Additionally, mathematical models of modern financial instruments have had a direct and significant influence on finance practice.

Early History

The origins of much of the mathematics in financial models traces to Louis Bachelier’s 1900 dissertation on the theory of speculation in the Paris markets. Completed at the Sorbonne in 1900, this work marks the twin births of both the continuous time mathematics of stochastic processes and the continuous time economics of option pricing. While analyzing option pricing, Bachelier provided two different derivations of the partial differential equation for the probability density for the Wiener process or Brownian motion. In one of the derivations, he works out what is now called the Chapman-Kolmogorov convolution probability integral. Along the way, Bachelier derived the method of reflection to solve for the probability function of a diffusion process with an absorbing barrier. Not a bad performance for a thesis on which the first reader, Henri Poincaré, gave less than a top mark! After Bachelier, option pricing theory laid dormant in the economics literature for over half a century until economists and mathematicians renewed study of it in the late 1960s. Jarrow and Protter [7] speculate that this may have been because the Paris mathematical elite scorned economics
as an application of mathematics.

Bachelier’s work was 5 years before Albert Einstein’s 1905 discovery of the same equations for his famous mathematical theory of Brownian motion. The editor of Annalen der Physik received Einstein’s paper on Brownian motion on May 11, 1905. The paper appeared later that year. Einstein proposed a model for the motion of small particles with diameters on the order of 0.001 mm suspended in a liquid. He predicted that the particles would undergo microscopically observable and statistically predictable motion. The English botanist Robert Brown had already reported such motion in 1827 while observing pollen grains in water with a microscope. The physical motion is now called Brownian motion in honor of Brown’s description.

Einstein calculated a diffusion constant to govern the rate of motion of suspended particles. The paper was Einstein’s justification of the molecular and atomic nature of matter. Surprisingly, even in 1905 the scientific community did not completely accept the atomic theory of matter. In 1908, the experimental physicist Jean-Baptiste Perrin conducted a series of experiments that empirically verified Einstein’s theory. Perrin thereby determined the physical constant known as Avogadro’s number for which he won the Nobel prize in 1926. Nevertheless, Einstein’s theory was difficult to rigorously justify mathematically. In a series of papers from 1918 to 1923, the mathematician Norbert Wiener constructed a mathematical model of Brownian motion. Wiener and others proved many surprising facts about his mathematical model of Brownian motion, research that continues today. In recognition of his work, his mathematical construction is often called the Wiener process. [7]

Growth of Mathematical Finance

Modern mathematical finance theory begins in the 1960s. In 1965 the economist Paul Samuelson published two papers that argue that stock prices fluctuate randomly [7]. One explained the Samuelson and Fama efficient markets hypothesis that in a well-functioning and informed capital market, asset-price dynamics are described by a model in which the best estimate of an asset’s future price is the current price (possibly adjusted for a fair expected rate of return.) Under this hypothesis, attempts to use past price data or publicly available forecasts about economic fundamentals to predict security prices cannot succeed. In the other paper with mathematician Henry McKean, Samuelson shows that a good model for stock price movements
is Geometric Brownian Motion. Samuelson noted that Bachelier’s model failed to ensure that stock prices would always be positive, whereas geometric Brownian motion avoids this error [7].

The most important development was the 1973 Black-Scholes model for option pricing. The two economists Fischer Black and Myron Scholes (and simultaneously, and somewhat independently, the economist Robert Merton) deduced an equation that provided the first strictly quantitative model for calculating the prices of options. The key variable is the volatility of the underlying asset. These equations standardized the pricing of derivatives in exclusively quantitative terms. The formal press release from the Royal Swedish Academy of Sciences announcing the 1997 Nobel Prize in Economics states that they gave the honor “for a new method to determine the value of derivatives. Robert C. Merton and Myron S. Scholes have, in collaboration with the late Fischer Black developed a pioneering formula for the valuation of stock options. Their methodology has paved the way for economic valuations in many areas. It has also generated new types of financial instruments and facilitated more efficient risk management in society.”

The Chicago Board Options Exchange (CBOE) began publicly trading options in the United States in April 1973, a month before the official publication of the Black-Scholes model. By 1975, traders on the CBOE were using the model to both price and hedge their options positions. In fact, Texas Instruments created a hand-held calculator specially programmed to produce Black-Scholes option prices and hedge ratios.

The basic insight underlying the Black-Scholes model is that a dynamic portfolio trading strategy in the stock can replicate the returns from an option on that stock. This is “hedging an option” and it is the most important idea underlying the Black-Scholes-Merton approach. Much of the rest of the book will explain what that insight means and how to apply it to calculate option values.

The story of the development of the Black-Scholes-Merton option pricing model is that Black started working on this problem by himself in the late 1960s. His idea was to apply the capital asset pricing model to value the option in a continuous time setting. Using this idea, the option value satisfies a partial differential equation. Black could not find the solution to the equation. He then teamed up with Myron Scholes who had been thinking about similar problems. Together, they solved the partial differential equation using a combination of economic intuition and earlier pricing formulas.

At this time, Myron Scholes was at MIT. So was Robert Merton, who
was applying his mathematical skills to problems in finance. Merton showed Black and Scholes how to derive their differential equation differently. Merton was the first to call the solution the Black-Scholes option pricing formula. Merton’s derivation used the continuous time construction of a perfectly hedged portfolio involving the stock and the call option together with the notion that no arbitrage opportunities exist. This is the approach we will take. In the late 1970s and early 1980s mathematicians Harrison, Kreps and Pliska showed that a more abstract formulation of the solution as a mathematical model called a martingale provides greater generality.

By the 1980s, the adoption of finance theory models into practice was nearly immediate. Additionally, the mathematical models used in financial practice became as sophisticated as any found in academic financial research [9].

Several explanations account for the different adoption rates of mathematical models into financial practice during the 1960s, 1970s and 1980s. Money and capital markets in the United States exhibited historically low volatility in the 1960s; the stock market rose steadily, interest rates were relatively stable, and exchange rates were fixed. Such simple markets provided little incentive for investors to adopt new financial technology. In sharp contrast, the 1970s experienced several events that led to market change and increasing volatility. The most important of these was the shift from fixed to floating currency exchange rates; the world oil price crisis resulting from the creation of the Middle East cartel; the decline of the United States stock market in 1973-1974 which was larger in real terms than any comparable period in the Great Depression; and double-digit inflation and interest rates in the United States. In this environment, the old rules of thumb and simple regression models were inadequate for making investment decisions and managing risk [9].

During the 1970s, newly created derivative-security exchanges traded listed options on stocks, futures on major currencies and futures on U.S. Treasury bills and bonds. The success of these markets partly resulted from increased demand for managing risks in a volatile economic market. This success strongly affected the speed of adoption of quantitative financial models. For example, experienced traders in the over the counter market succeeded by using heuristic rules for valuing options and judging risk exposure. However, these rules of thumb were inadequate for trading in the fast-paced exchange-listed options market with its smaller price spreads, larger trading volume and requirements for rapid trading decisions while monitoring prices in both
the stock and options markets. In contrast, mathematical models like the Black-Scholes model were ideally suited for application in this new trading environment \[9\].

The growth in sophisticated mathematical models and their adoption into financial practice accelerated during the 1980s in parallel with the extraordinary growth in financial innovation. A wave of de-regulation in the financial sector was an important element driving innovation.

Conceptual breakthroughs in finance theory in the 1980s were fewer and less fundamental than in the 1960s and 1970s, but the research resources devoted to the development of mathematical models was considerably larger. Major developments in computing power, including the personal computer and increases in computer speed and memory enabled new financial markets and expansions in the size of existing ones. These same technologies made the numerical solution of complex models possible. Faster computers also speeded up the solution of existing models to allow virtually real-time calculations of prices and hedge ratios.

**Ethical considerations**

According to M. Poovey \[11\], Enron developed new derivatives specifically to take advantage of de-regulation. Poovey says that derivatives remain largely unregulated, for they are too large, too virtual, and too complex for industry oversight to police. In 1997-1998 the Financial Accounting Standards Board (an industry standards organization whose mission is to establish and improve standards of financial accounting) did try to rewrite the rules governing the recording of derivatives, but ultimately they failed: in the 1999-2000 session of Congress, lobbyists for the accounting industry persuaded Congress to pass the Commodities Futures Modernization Act, which exempted or excluded “over the counter” derivatives from regulation by the Commodity Futures Trading Commission, the federal agency that monitors the futures exchanges. Current law requires only banks and other financial institutions to reveal their derivatives positions. In contrast, Enron, originally an energy and commodities firm which collapsed in 2001 due to an accounting scandal, never registered as a financial institution and was never required to disclose the extent of its derivatives trading.

In 1995, the sector composed of finance, insurance, and real estate overtook the manufacturing sector in America’s gross domestic product. By the year 2000 this sector led manufacturing in profits. The Bank for In-
ternational Settlements estimates that in 2001 the total value of derivative contracts traded approached one hundred trillion dollars, which is approximately the value of the total global manufacturing production for the last millennium. In fact, one reason that derivatives trades have to be electronic instead of involving exchanges of capital is that the amounts exceed the total of the world’s physical currencies.

Prior to the 1970s, mathematical models had a limited influence on finance practice. But since 1973 these models have become central in markets around the world. In the future, mathematical models are likely to have an indispensable role in the functioning of the global financial system including regulatory and accounting activities.

We need to seriously question the assumptions that make models of derivatives work: the assumptions that the market follows probability models and the assumptions underneath the mathematical equations. But what if markets are too complex for mathematical models? What if unprecedented events do occur, and when they do – as we know they do – what if they affect markets in ways that no mathematical model can predict? What if the regularity that all mathematical models assume ignores social and cultural variables that are not subject to mathematical analysis? Or what if the mathematical models traders use to price futures actually influence the future in ways that the models cannot predict and the analysts cannot govern?

Any virtue can become a vice if taken to extreme, and just so with the application of mathematical models in finance practice. At times, the mathematics of the models becomes too interesting and we lose sight of the models’ ultimate purpose. Futures and derivatives trading depends on the belief that the stock market behaves in a statistically predictable way; in other words, that probability distributions accurately describe the market. The mathematics is precise, but the models are not, being only approximations to the complex, real world. The practitioner should apply the models only tentatively, assessing their limitations carefully in each application. The belief that the market is statistically predictable drives the mathematical refinement, and this belief inspires derivative trading to escalate in volume every year.

Financial events since late 2008 show that the concerns of the previous paragraphs have occurred. In 2009, Congress and the Treasury Department considered new regulations on derivatives markets. Complex derivatives called credit default swaps appear to have used faulty assumptions that did not account for irrational and unprecedented events, as well as social
and cultural variables that encouraged unsustainable borrowing and debt. Extremely large positions in derivatives which failed to account for unlikely events caused bankruptcy for financial firms such as Lehman Brothers and the collapse of insurance giants like AIG. The causes are complex, but critics fix some of the blame on the complex mathematical models and the people who created them. This blame results from distrust of that which is not understood. Understanding the models and their limitations is a prerequisite for creating a future which allows proper risk management.

Sources


Problems to Work for Understanding

1. Write a short summary of the “tulip mania” in seventeenth century Holland.
2. Write a short summary of the “South Sea Island” bubble in eighteenth century England.

3. Pick a commodity and find current futures prices for that commodity.

4. Pick a stock and find current options prices on that stock.

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**Reading Suggestion:**

**References**


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**Outside Readings and Links:**


2. [Clip from “The Trillion Dollar Bet”](#) Accessed Fri Jul 24, 2009 5:29 AM.

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