Stochastic Processes and Advanced Mathematical Finance

A Model of Collateralized Debt Obligations

Rating

Mathematically Mature: may contain mathematics beyond calculus with proofs.
Section Starter Question

How can you evaluate cumulative binomial probabilities

\[ P \left[ S_N \leq n \right] = \sum_{j=0}^{n} \binom{N}{j} p^j (1 - p)^{N-j} \]

when the value of \( N \) is large, say \( N = 100 \) and the value of \( p \) is small, say \( p = 0.05 \)?

Key Concepts

1. We can make a simple mathematical model of a type of financial derivative using only the idea of a binomial probability.

2. We must investigate the sensitivity of the model to the parameter values to completely understand the model.

3. This simple model provides our first illustration of the modeling cycle in mathematical finance, but even so, it yields valuable insights.

Vocabulary

1. A tranch is a portion or slice of a set of securities. The common use of tranch is an issue of bonds, often derived from mortgages, with each tranch distinguished from other tranches by maturity or rate of return.
2. A **collateralized debt obligation** or **CDO** is a derivative security backed by a pool or slice of other securities. CDOs can be made of any kind of debt and do not necessarily derive from mortgages. Securities or bonds derived from mortgages are more specifically called Collateralized Mortgage Obligations or CMOs or even more specifically RMBS for “residential mortgage backed securities”. The terms are often used interchangeably but CDO is the most common. CDOs contain multiple slices or tranches, each slice made of debt with an associated risk. Investors who want exposure to the income generated by the debt but do not want to purchase the debt itself buy the CDOs.

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**Mathematical Ideas**

**A binomial model of mortgages**

To illustrate the previous sections altogether we will make a simple binomial probability model of a financial instrument called a CDO, standing for “Collateralized Debt Obligation”. The market in this derivative financial instrument is large, in 2007 amounting to at least $1.3 trillion dollars, of which 56% came from derivatives based on residential mortgages. Heavy reliance on these financial derivatives based on the real estate market contributed to the demise of some old-line brokerage firms such as Bear Stearns and Merrill Lynch in the autumn of 2008. The quick loss in value of these derivatives sparked a lack of economic confidence which led to the sharp economic downturn in the fall of 2008 and the subsequent recession. We will build a “stick-figure” model of these instruments, and even this simple model will demonstrate that the CDOs were far more sensitive to mortgage failure rates than was commonly understood. While this model does not fully describe CDOs, it does provide an interesting and accessible example of the modeling process in mathematical finance.

Consider the following financial situation. A lending company has made 100 mortgage loans to home-buyers. We make two modeling assumptions
about the loans.

1. For simplicity, each loan will have precisely one of 2 outcomes. Either the home-buyer will pay off the loan resulting in a profit of 1 unit of money to the lender, or the home-buyer will default on the loan, resulting in a payoff or profit to the company of 0. For further simplicity we will say that the unit profit is $1. (The payoff is typically in the thousands of dollars.)

2. We assume that the probability of default on a loan is $p$ and we will assume that the probability of default on each loan is independent of default on all the other loans.

Let $S_{100}$ be the number of loans that default, resulting in a total profit of $100 - S_{100}$. The probability of $n$ or fewer of these 100 mortgage loans defaulting is

$$
\mathbb{P}\left[S_{100} \leq n\right] = \sum_{j=0}^{n} \binom{100}{j} p^j (1 - p)^{100-j}.
$$

We can evaluate this expression in several ways including direct calculation and approximation methods. For our purposes here, one can use a binomial probability table, or more easily a computer program which has a cumulative binomial probability function. The expected number of defaults is $100p$, the resulting expected loss is $100p$ and the expected profit is $100(1 - p)$.

But instead of simply making the loans and waiting for them to be paid off the loan company wishes to bundle these debt obligations differently and sell them as a financial derivative contract to an investor. Specifically, the loan company will create a collection of 100 contracts also called tranches. Contract 1 will pay 1 dollar if 0 of the loans default. Contract 2 will pay 1 dollar if 0 or 1 of the loans defaults, and in general contract $n$ will pay 1 dollar if $n - 1$ or fewer of the loans defaults. (This construction is a much simplified model of mortgage backed securities. In actual practice banks combine many mortgages with various levels of risk and then package them into derivative securities with differing levels of risk called tranches. Each tranche pays out a revenue stream, not a single unit payment. A tranche is usually backed by thousands of mortgages.)

Suppose to be explicit that 5 of the 100 loans defaults. Then the seller will have to pay off contracts 6 through 101. The loan company who creates the contracts will receive 95 dollars from the 95 loans that do not default.
and will pay out 95 dollars. If the seller prices the contracts appropriately, then the lender will have enough money to cover the payout and will have some profit from selling the contracts.

For the contract buyer, the contract will either pay off with a value of 1 or will default. The probability of payoff on contract $i$ will be the sum of the probabilities that $i - 1$ or fewer mortgages default:

$$\sum_{j=0}^{i-1} \binom{100}{j} p^j (1-p)^{100-j},$$

that is, a binomial cumulative distribution function. The probability of default on contract $i$ will be a binomial complementary distribution function, which we will denote by

$$p_T(i) = 1 - \sum_{j=0}^{i-1} \binom{100}{j} p^j (1-p)^{100-j}.$$

We should calculate a few default probabilities: The probability of default on contract 1 is the probability of 0 defaults among the 100 loans,

$$p_T(1) = 1 - \binom{100}{0} p^0 (1-p)^{100} = 1 - (1-p)^{100}.$$  

If $p = 0.05$, then the probability of default is 0.99408. But for the contract 10, the probability of default is 0.028188. By the 10th contract, this financial construct has created an instrument that is safer than owning one of the original mortgages! Because the newly derived security combines the risks of several individual loans, under the assumptions of the model it is less exposed to the potential problems of any one borrower.

The expected payout from the collection of contracts will be

$$\mathbb{E}[U] = \sum_{n=0}^{100} \sum_{j=0}^{n} \binom{100}{j} p^j (1-p)^{100-j} = \sum_{j=0}^{100} \binom{100}{j} p^j (1-p)^{100-j} = 100p.$$  

That is, the expected payout from the collection of contracts is exactly the same as the expected payout from the original collection of mortgages. However, the lender will also receive the profit of the contracts sold. Moreover, since the lender is now only selling the possibility of a payout derived from
mortgages and not the mortgages themselves, the lender can even sell the
same contract several times to several different buyers if the profits outweigh
the risk of multiple payouts.

Why rebundle and sell mortgages as tranches? The reason is that for
many of the tranches the risk exposure is less, but the payout is the same
as owning a mortgage loan. Reduction of risk with the same payout is very
desirable for many investors. Those investors may even pay a premium for
low risk investments. In fact, some investors like pension funds are required
by law, regulation or charter to invest in securities that have a low risk. Some
investors may not have direct access to the mortgage market, again by law,
regulation or charter, but in a rising (or bubble) market they desire to get
into that market. These derivative instruments look like a good investment
to them.

Collateralized Debt Obligations

If rebundling mortgages once is good, then doing it again should be better!
So now assume that the loan company has 10,000 loans, and that it divides
these into 100 groups of 100 each, and creates contracts as above. Label the
groups Group 1, Group 2, and so on to Group 100. Now for each group, the
bank makes 100 new contracts. Contract 1.1 will pay off $1 if 0 mortgages
in Group 1 default. Contract 1.2 will pay $1 if 0 or 1 mortgages in Group 1
default. Continue, so for example, Contract 1.10 will pay $1 if 0, 1, 2, . . . , 9
mortgages default in Group 1. Do this for each group, so for example, in the
last group, Contract 100.10 will pay off $1 if 0, 1, 2, . . . , 9 mortgages in Group
100 default. Now for example, the lender gathers up the 100 contracts j.10,
one from each group, into a secondary group and bundles them just as before,
paying off 1 dollar if i − 1 or fewer of these contracts j.10 defaults. These
new derivative contracts are now called collateralized debt obligations or
CDOs. Again, this is a much simplified “stick-figure” model of a real CDO,
see [2]. Sometimes, these second-level constructs are called a “CDO squared”
[1]. Just as before, the probability of payout for the contracts j.10 is

\[
\sum_{j=0}^{9} \binom{100}{j} p_T(10)^j (1 - p_T(10))^{100-j}
\]
and the probability of default is
\[ p_{CDO}(10) = 1 - \sum_{j=0}^{9} \binom{100}{j} p_T(10)^j (1 - p_T(10))^{100-j}. \]

For example, with \( p = 0.05 \) and \( p_T(10) = 0.028188 \) then \( p_{CDO}(10) = 0.00054385 \). Roughly, the CDO has only 1/100 of the default probability of the original mortgages, by virtue of re-distributing the risk.

The construction of the contracts \( j.10 \) is a convenient example to illustrate the relative values of the risk of the original mortgage, the tranche and the second-order tranche or “CDO-squared”. The number 10 for the contracts \( j.10 \) is not special. In fact, the bank could make a “super-CDO \( j.M.N \)” where it pays $1 if 0, 1, 2, 3 \ldots, M - 1 of the \( N \)-tranches in group \( j \) fail, even without a natural relationship between \( M \) and \( N \). Even more is possible, the bank could make a contract that would pay some amount if some arbitrary finite sequence of contracts composed from some arbitrary sequence of tranches from some set of groups fails. We could still calculate the probability of default or non-payment, it’s just a mathematical problem. The only questions would be what contracts would be risky, what the bank could sell and how to price everything to be profitable to the bank.

The possibility of creating this “super-CDO \( j.M.N \)” illustrates one problem with the whole idea of CDOs that led to the collapse and the recession of 2008. These contracts quickly become confusing and hard to understand. These contracts are now so far removed from the reality of a homeowner paying a mortgage to a bank that they become their own gambling game. But the next section analyzes a more serious problem with these second-order contracts.

**Sensitivity to the parameters**

Now we investigate the robustness of the model. We do this by varying the probability of mortgage default to see how it affects the risk of the tranches and the CDOs.

Assume that the underlying mortgages actually have a default probability of 0.06, a 20% increase in the risk although it is only a 0.01 increase in the actual rates. This change in the default rate may be due to several factors. One may be the inherent inability to measure a fairly subjective parameter such as “mortgage default rate” accurately. Finding the probability
of a home-owner defaulting is not the same as calculating a losing bet in a dice game. Another may be a faulty evaluation (usually over confident or optimistic) of the default rates themselves by the agencies who provide the service of evaluating the risk on these kinds of instruments. Some economic commentators allege that before the 2008 economic crisis the rating agencies were under intense competitive pressure to provide “good” ratings in order to get the business of the firms who create derivative instruments. The agencies may have shaded their ratings to the favorable side in order to keep the business. Finally, the underlying economic climate may be changing and the earlier estimate, while reasonable for the prior conditions, is no longer valid. If the economy deteriorates or the jobless rate increases, weak mortgages called sub-prime mortgages may default at increased rates.

Now with \( p = 0.06 \) we calculate that each contract \( j.10 \) has a default probability of 0.078, a 275% increase from the previous probability of 0.028. Worse, the 10th CDO-squared made of the contracts \( j.10 \) will have a default probability of 0.247, an increase of over 45,400%! The financial derivatives amplify any error in measuring the default rate to a completely unacceptable risk. The model shows that the financial instruments are not robust to errors in the assumptions!

But shouldn’t the companies either buying or selling the derivatives recognize this? A human tendency is to blame failures, including the failures of the Wall Street giants, on ignorance, incompetence or wrongful behavior. In this case, the traders and “rocket scientists” who created the CDOs were probably neither ignorant nor incompetent. Because they ruined a profitable endeavor for themselves, we can probably rule out malfeasance too. But distraction resulting from an intense competitive environment allowing no time for rational reflection along with overconfidence during a bubble can make us willfully ignorant of the conditions. A failure to properly complete the modeling cycle leads the users to ignore the real risks.

**Criticism of the model**

This model is far too simple to base any investment strategy or more serious economic analysis on it. First, a binary outcome of either pay-off or default is too simple. Lenders will restructure shaky loans or they will sell them to other financial institutions so that the lenders will probably get some return, even if less than originally intended.

The assumption of a uniform probability of default is too simple by far.
Lenders make some loans to safe and reliable home-owners who dutifully pay off the mortgage in good order. Lenders also make some questionable loans to people with poor credit ratings, these are called sub-prime loans or sub-prime mortgages. The probability of default is not the same. In fact, rating agencies grade mortgages and loans according to risk. There are 20 grades ranging from AAA with a 1-year default probability of less than 0.001 through BBB with a 1-year default probability of slightly less than 0.01 to CC with a 1-year default probability of more than 0.35. The mortgages may also change their rating over time as economic conditions change, and that will affect the derived securities. Also too simple is the assumption of an equal unit payoff for each loan.

The assumption of independence is clearly incorrect. The similarity of the mortgages increases the likelihood that most will prosper or suffer together and potentially default together. Due to external economic conditions, such as an increase in the unemployment rate or a downturn in the economy, default on one loan may indicate greater probability of default on other, even geographically separate loans, especially sub-prime loans. This is the most serious objection to the model, since it invalidates the use of binomial
probabilities.

However, relaxing any assumptions make the calculations much more difficult. The non-uniform probabilities and the lack of independence means that elementary theoretical tools from probability are not enough to analyze the model. Instead, simulation models will be the next means of analysis.

Nevertheless, the sensitivity of the simple model should make us very wary of optimistic claims about the more complicated model.

Sources

This section is adapted from a presentation by Jonathan Kaplan of D.E. Shaw and Co. in summer 2010. Allan Baktoft Jakobsen of Copenhagen suggested some improvements and clarifications in December 2013. The definitions are derived from definitions at investorwords.com. The definition of CDO squared is noted in [1, page 166]. Some facts and figures are derived from the graphics at Portfolio.com: What’s a CDO [3] and Wall Street Journal.com: The Making of a Mortgage CDO [2]

Problems to Work for Understanding

1. Suppose a 20% decrease in the default probability from 0.05 to 0.04 occurs. By what factor do the default rates of the 10-tranches and the derived 10th CDO change?

2. For the tranches create a table of probabilities of default for contracts $i = 5$ to $i = 15$ for probabilities of default $p = 0.03, 0.04, 0.05, 0.06$ and $0.07$ and determine where the contracts become safer investments than the individual mortgages on which they are based.

3. For a base mortgage default rate of 5%, draw the graph of the default rate of the contracts as a function of the contract number.
4. The text asserts that the expected payout from the collection of contracts will be

\[
\mathbb{E}[U] = \sum_{n=0}^{100} \sum_{j=0}^{n} \binom{100}{j} p^j (1-p)^{100-j} = \sum_{j=0}^{100} j \binom{100}{j} p^j (1-p)^{100-j} = 100p.
\]

That is, the expected payout from the collection of contracts is exactly the same the expected payout from the original collection of mortgages. More generally, show that

\[
\sum_{n=0}^{N} \sum_{j=0}^{n} a_j = \sum_{j=0}^{N} j \cdot a_j.
\]

5. Write the general expressions for the probabilities of payout and default for the \(i\)th contract from the CDO-squared.

6. The following problem does not have anything to do with money, mortgages, tranches, or finance. It is instead a problem that creates and investigates a mathematical model using binomial probabilities, so it naturally belongs in this section. This problem is adapted from the classic 1943 paper by Robert Dorfman on group blood testing for syphilis among US military draftees.

Suppose that you have a large population that you wish to test for a certain characteristic in their blood or urine (for example, testing athletes for steroid use or military personnel for a particular disease). Each test will be either positive or negative. Since the number of individuals to be tested is quite large, we can expect that the cost of testing will also be large. How can we reduce the number of tests needed and thereby reduce the costs? If the blood could be pooled by putting a portion of, say, 10 samples together and then testing the pooled sample, the number of tests might be reduced. If, however, the pooled sample is positive, we only know that at least one of the individuals in the sample will test positive. Each member of the sample must then be retested individually and a total of 11 tests will be necessary to do the job. The larger the group size, the more we can eliminate with one test, but the more likely the group is to test positive. If the
blood could be pooled by putting $G$ samples together and then testing
the pooled sample, the number of tests required might be minimized.
Create a model for the blood testing cost involving the probability of
an individual testing positive ($p$) and the group size ($G$) and use the
model to minimize the total number of tests required. Investigate the
sensitivity of the cost to the probability $p$.

7. The following problem does not have anything to do with money, mort-
gages, tranches, or finance. It is instead a problem that creates and
investigates a mathematical model using binomial probabilities, so it
naturally belongs in this section.

Suppose you are taking a test with 25 multiple-choice questions. Each
question has 5 choices. Each problem is scored so that a correct an-
swer is worth 6 points, an incorrect answer is worth 0 points, and an
unanswered question is worth 1.5 points. You wish to score at least
100 points out of the possible 150 points. The goal is to create a model
of random guessing that optimizes your chances of achieving a score of
at least 100.

(a) How many questions must you answer correctly, leaving all other
questions blank, to score your goal of 100 points? For reference,
let this number of questions be $N$.

(b) Discuss the form of your mathematical model for the number of
questions required for success. What mathematics did you use for
the solution?

(c) Suppose you can only answer $N-1$ questions. Create and analyze
a model of guessing on unanswered questions to determine the
optimal number of questions to guess.

(d) Discuss the form of your mathematical model for success with
guessing. What mathematics did you use for the solution?

(e) Now begin some testing and sensitivity analysis. Suppose you can
only answer $N-i$ questions, where $1 \leq i \leq N$. Adjust and analyze
your model of guessing on unanswered questions to determine the
optimal number of questions to guess.

(f) Test the sensitivity of the model to changes in the probability of
guessing a correct answer.
(g) Critically analyze the model, interpreting it in view of the sensitivity analysis. Change the model appropriately.

Reading Suggestion:

References


Outside Readings and Links:


2. Portfolio.com: What’s a CDO Another animated graphic explanation from Portfolio.com describing mortgage backed debt obligations.
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