Stochastic Processes and Advanced Mathematical Finance

Arbitrage

Rating

Student: contains scenes of mild algebra or calculus that may require guidance.
Section Starter Question

It’s the day of the big game. You know that your rich neighbor really wants to buy tickets, in fact you know he’s willing to pay $50 a ticket. While on campus, you see a hand lettered sign offering “two general-admission tickets at $25 each, inquire immediately at the mathematics department”. You have your phone with you, what should you do? Discuss whether this is a frequent event, and why or why not? Is this market efficient? Is there any risk in this market?

Key Concepts

1. An **arbitrage opportunity** is a circumstance where the simultaneous purchase and sale of related securities is guaranteed to produce a risk-less profit. Arbitrage opportunities should be rare, but in a world-wide market they can occur.

2. Prices change as the investors move to take advantage of such an opportunity. As a consequence, the arbitrage opportunity disappears. This becomes an economic principle: *in an efficient market there are no arbitrage opportunities.*

3. The principle of **arbitrage pricing** is that any two investments with identical payout streams must have the same price.
Vocabulary

1. **Arbitrage** is locking in a riskless profit by simultaneously entering into transactions in two or more markets, exploiting mismatches in pricing.

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Mathematical Ideas

**Definition of Arbitrage**

The notion of arbitrage is crucial in the modern theory of finance. It is a cornerstone of the Black, Scholes and Merton option pricing theory, developed in 1973, for which Scholes and Merton received the Nobel Prize in 1997 (Fisher Black died in 1995).

An **arbitrage opportunity** is a circumstance where the simultaneous purchase and sale of related securities is guaranteed to produce a riskless profit. Arbitrage opportunities should be rare, but in a world-wide basis some do occur.

This section illustrates the concept of arbitrage with simple examples.

**An arbitrage opportunity in exchange rates**

Consider a stock that is traded in both New York and London. Suppose that the stock price is $172 in New York and £100 in London at a time when the exchange rate is $1.7500 per pound. An arbitrageur in New York could simultaneously buy 100 shares of the stock in New York and sell them in London to obtain a risk-free profit of

\[
100 \text{ shares} \times 100 \text{ £/share} \times 1.75 \text{ $/£} - 100 \text{ shares} \times 172 \text{ $/share} = 300
\]

in the absence of transaction costs. Transaction costs would probably eliminate the profit on a small transaction like this. However, large investment houses face low transaction costs in both the stock market and the foreign
exchange market. Trading firms would find this arbitrage opportunity very attractive and would try to take advantage of it in quantities of many thousands of shares.

The shares in New York are underpriced relative to the shares in London with the exchange rate taken into consideration. However, note that the demand for the purchase of many shares in New York would soon drive the price up. The sale of many shares in London would soon drive the price down. The market would soon reach a point where the arbitrage opportunity disappears.

An arbitrage opportunity in gold contracts

Suppose that the current market price (called the spot price) of an ounce of gold is $398 and that an agreement to buy gold in three months time would set the price at $390 per ounce (called a forward contract). Suppose that the price for borrowing gold (actually the annualized 3-month interest rate for borrowing gold, called the convenience price) is 10%. Additionally assume that the annualized interest rate on 3-month deposits (such as a certificate of deposit at a bank) is 4%. This set of economic circumstances creates an arbitrage opportunity. The arbitrageur can borrow one ounce of gold, immediately sell the borrowed gold at its current price of $398 (this is called shorting the gold), lend this money out for three months and simultaneously enter into the forward contract to buy one ounce of gold at $390 in 3 months. The cost of borrowing the ounce of gold is

\[ \$398 \times 0.10 \times \frac{1}{4} = \$9.95 \]

and the interest on the 3-month deposit amounts to

\[ \$398 \times 0.04 \times \frac{1}{4} = \$3.98. \]

The investor will therefore have 398.00 + 3.98 – 9.95 = 392.03 in the bank account after 3 months. Purchasing an ounce of gold in 3 months, at the forward price of $390 and immediately returning the borrowed gold, he will make a profit of $2.03. This example ignores transaction costs and assumes interests are paid at the end of the lending period. Transaction costs would probably consume the profits in this one-ounce example. However, large-volume gold-trading arbitrageurs with low transaction costs would take advantage of this opportunity by purchasing many ounces of gold.
Figure 1: A schematic diagram of the cash flow in the gold arbitrage example.

Figure 1 schematically diagrams this transaction. Time is on the horizontal axis, and cash flow is vertical, with the arrow up if cash comes in to the investor, and the arrow down if cash flows out from the investor.

**Discussion about arbitrage**

Arbitrage opportunities as just described cannot last for long. In the first example, as arbitrageurs buy the stock in New York, the forces of supply and demand will cause the New York dollar price to rise. Similarly as the arbitrageurs sell the stock in London, they drive down the London sterling price. The two stock prices will quickly become equal at the current exchange rate. Indeed the existence of profit-hungry arbitrageurs (usually pictured as frenzied traders carrying on several conversations at once!) makes it unlikely that a major disparity between the sterling price and the dollar price could ever exist in the first place. In the second example, once arbitrageurs start to sell gold at the current price of $398, the price will drop. The demand for the 3-month forward contracts at $390 will cause the price to rise. Although arbitrage opportunities can arise in financial markets, they cannot last long.

Generalizing, the existence of arbitrageurs means that in practice, only tiny arbitrage opportunities are observed only for short times in most financial markets. As soon as sufficiently many observant investors find the arbitrage, the prices quickly change as the investors buy and sell to take advantage of such an opportunity. As a consequence, the arbitrage opportunity disappears. The principle can stated as follows: in an efficient market there are no arbitrage opportunities. In this text many arguments depend on the
assumption that arbitrage opportunities do not exist, or equivalently, that we are operating in an efficient market.

A joke illustrates this principle well: A mathematical economist and a financial analyst are walking down the street together. Suddenly each spots a $100 bill lying in the street at the curb! The financial analyst yells “Wow, a $100 bill, grab it quick!” The mathematical economist says “Don’t bother, if it were a real $100 bill, somebody would have picked it up already.” Arbitrage opportunities are like $100 bills on the ground, they do exist in real life, but one has to be quick and observant. For purposes of mathematical modeling, we can treat arbitrage opportunities as non-existent as $100 bills lying in the street. It might happen, but we don’t base our financial models on the expectation of finding them.

The principle of arbitrage pricing is that any two investments with identical payout streams must have the same price. If this were not so, we could simultaneously sell the more expensive instrument and buy the cheaper one; the payment from our sale exceeds the payment for our purchase. We can make an immediate profit.

Before the 1970s most economists approached the valuation of a security by considering the probability of the stock going up or down. Economists now find the price of a security by arbitrage without the consideration of probabilities. We will use the principle of arbitrage pricing extensively in this text.

Sources

Problems to Work for Understanding

1. Consider the hypothetical country of Elbonia, where the government has declared a “currency band” policy, in which the exchange rate between the domestic currency, the Elbonian Bongo Buck, denoted by EBB, and the U.S. Dollar must fluctuate in a prescribed band, namely:

\[ 0.95\text{USD} \leq \text{EBB} \leq 1.05\text{USD} \]

for at least one year. Suppose also that the Elbonian government has issued 1-year notes denominated in the EBB that pay a continuously compounded interest rate of 30%. Assuming that the corresponding continuously compounded interest rate for US deposits is 6%, show that an arbitrage opportunity exists. (Adapted from *Quantitative Modeling of Derivative Securities*, by M. Avellaneda and P. Laurence, Chapman and Hall, Boca Raton, 2000, Exercises 1.7.1, page 18).

2. (a) At a certain time, the exchange rate between the U.S. Dollar and the Euro was 1.4280, that is, it cost $1.4280 to buy one Euro. At that time, the 1-year Fed Funds rate, (the bank-to-bank lending rate), in the United States was 4.7500% (assume it is compounded continuously). The forward rate (the exchange rate in a forward contract that allows you to buy Euros in a year) for purchasing Euros 1 year from today was 1.4312. What was the corresponding bank-to-bank lending rate in Europe (assume it is compounded continuously), and what principle allows you to claim that value?

(b) Find the current exchange rate between the U.S. Dollar and the Euro, the current 1-year Fed Funds rate, and the current forward rate for exchange of Euros to Dollars. Use those values to compute the bank-to-bank lending rate in Europe.

3. According to the article “Bullion bulls” on page 81 in the October 8, 2009 issue of *The Economist*, gold rose from about $510 per ounce in January 2006 to about $1050 per ounce in October 2009, 46 months later.

(a) What was the continuously compounded annual rate of increase of the price of gold over this period?
(b) In October 2009, one could borrow or lend money at 5% interest, again assume it was compounded continuously. In view of this, describe a strategy that would have made a profit in October 2010, involving borrowing or lending money, assuming that the rate of increase in the price of gold stayed constant over this time.

(c) The article suggests that the rate of increase for gold would stay constant. In view of this, what do you expect to happen to interest rates and what principle allows you to conclude that?

4. Consider a market that has a security and a bond so that money can be borrowed or loaned at an annual interest rate of \( r \) compounded continuously. At the end of a time period \( T \), the security will have increased in value by a factor \( U \) to \( SU \), or decreased in value by a factor \( D \) to value \( SD \). Show that a forward contract with strike price \( k \) that is, a contract to buy the security which has potential payoffs \( SU - k \) and \( SD - k \) should have the strike price set at \( S \exp(rT) \) to avoid an arbitrage opportunity.

Reading Suggestion:

References


Outside Readings and Links:

1. [A lecture on currency arbitrage](http://www.example.com) A link to a youtube video.

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