# Math 489/889

# Stochastic Processes and

# Advanced Mathematical Finance

# Homework 6

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Due Monday, October 13, 2010

## Problem 1

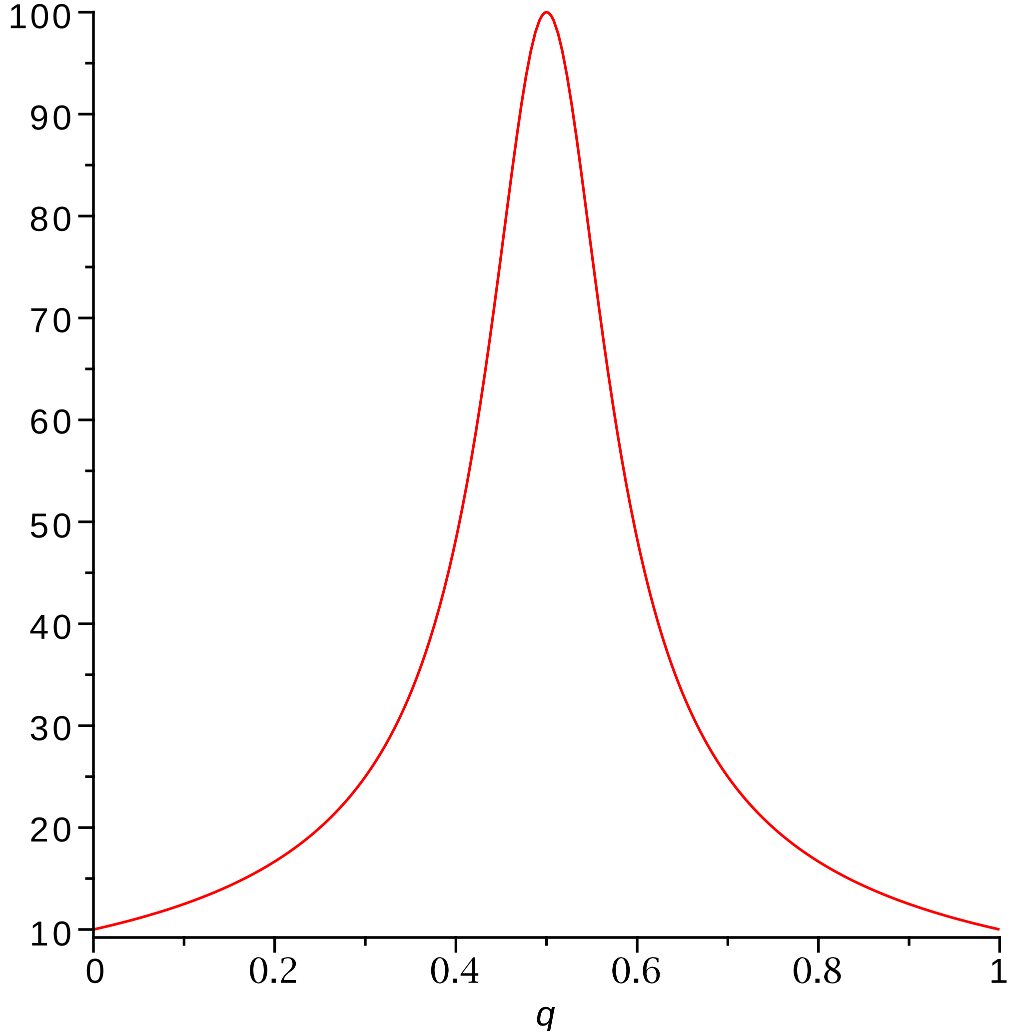
$$Duration := \frac{T0}{(q-p)} - \frac{a}{(q-p)} \cdot \frac{\left(1 - \left(\frac{q}{p}\right)^{T0}\right)}{\left(1-\left(\frac{q}{p}\right)^{a}\right)}$$

$$\frac{T0}{q-p} - \frac{a\left(1-\left(\frac{q}{p}\right)^{T0}\right)}{(q-p)\left(1-\left(\frac{q}{p}\right)^{a}\right)} \quad \textbf{(1.1)}$$

### Part a

For $ T_0 = 10 $  and $ a = 20 $ , draw a graph of theduration as a function of the probability $ q $ .

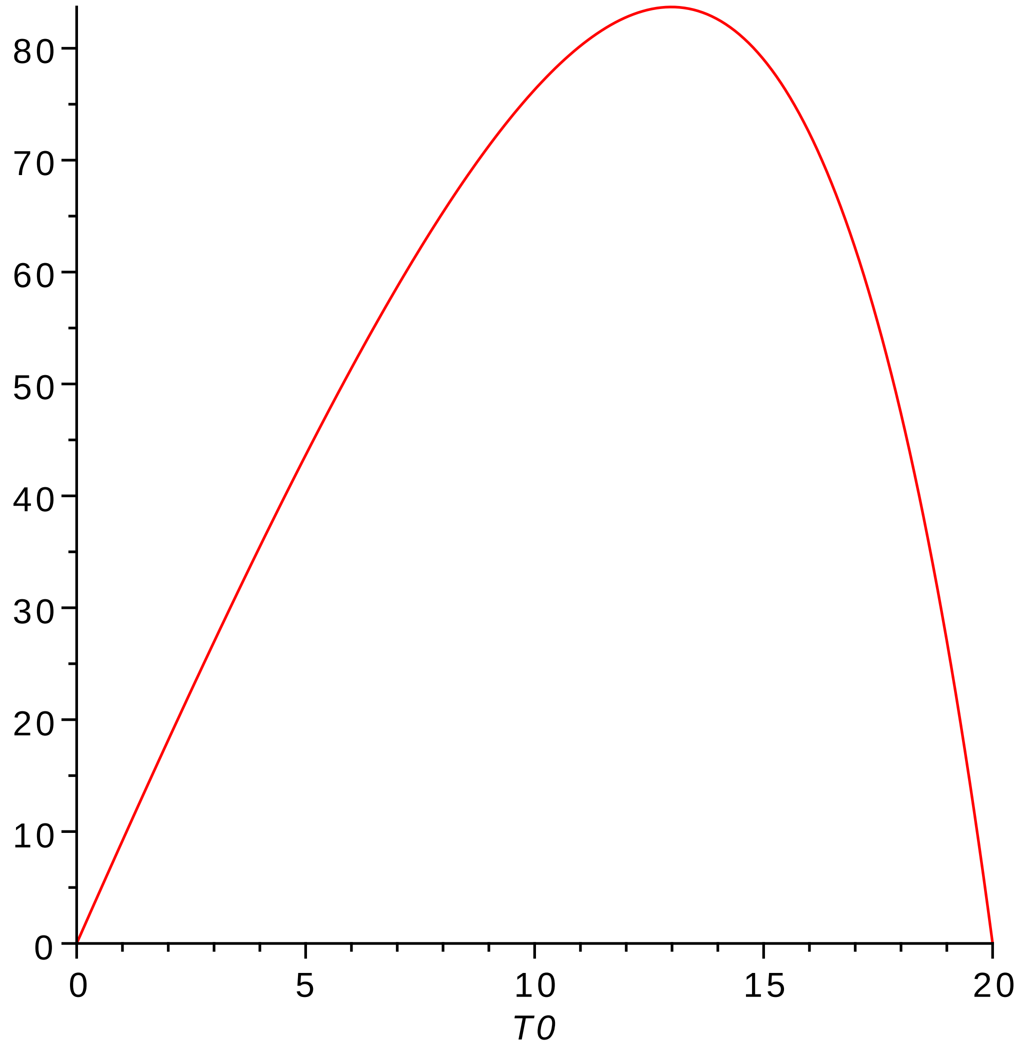
$plot(\ subs(\ \{T0 = 10, a = 20, p = 1 - q\}, Duration), q = 0\,..1)$




## Part b

For $ a = 20 $ and $ q = 0.55 $ draw a graph of the expected duration as a function of $ T_0 $ .

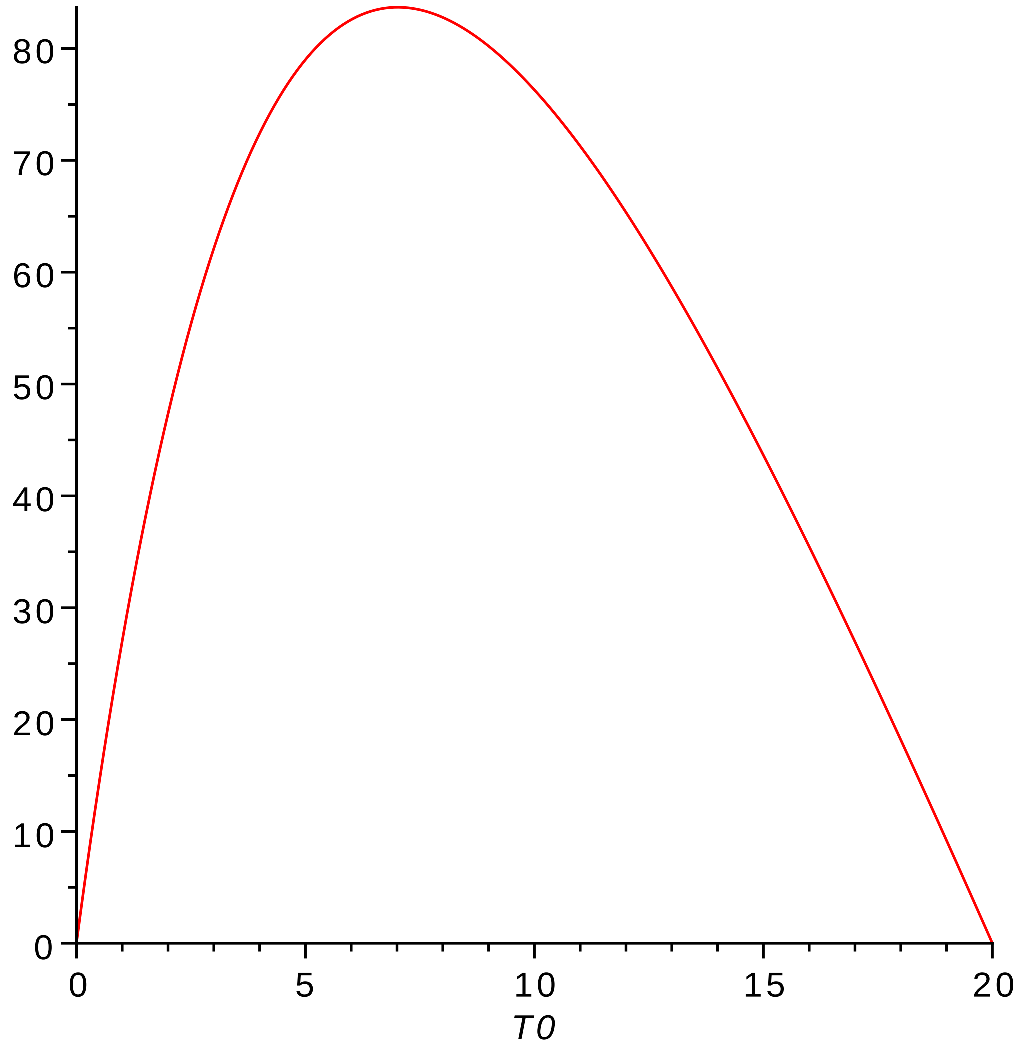
*plot*( *subs*( {*q* = 0.55, *a* = 20, *p* = 0.45}, *Duration*), *T0* = 0 ..20)




## Part c

For $ a = 20 $ and $ q = 0.45 $ draw a graph of the expected duration as a function of $ T_0 $ .

$plot(\ subs(\ \{q = 0.45,\ a = 20,\ p = 0.55\},\ Duration),\ T0 = 0\,..20)$




## Problem 2

The boundary condition at state 26 says that the duration of the "game" from 26 down to 18 is the sum of hte duration of the subsequent games from 26 down to 25, and then from 25 to 18. We can compute the duration of the first sub-game using Corollary 1. Then the set of 9 first-step equations in 9 unknowns is:

$$eqn26 := D26 = \frac{1}{\left(\frac{2}{3} - \frac{1}{3}\right)} + D25;$$

$$eqn25 := D25 = \left(\frac{1}{3}\right) \cdot D26 + \left(\frac{2}{3}\right) \cdot D24 + 1; \; eqn24 := D24 = \left(\frac{1}{3}\right) \cdot D25 + \left(\frac{2}{3}\right) \cdot D23 + 1; \; eqn23 := D23 = \left(\frac{1}{3}\right) \cdot D24 + \left(\frac{2}{3}\right) \cdot D22 + 1; \; eqn22 := D22 = \left(\frac{1}{3}\right) \cdot D23 + \left(\frac{2}{3}\right) \cdot D21 + 1;$$

$eqn21 := D21 = \left(\frac{9}{10}\right) \cdot D22 + \left(\frac{1}{10}\right) \cdot D20 + 1;$

$eqn20 := D20 = \left(\frac{3}{4}\right) \cdot D21 + \left(\frac{1}{4}\right) \cdot D19 + 1; eqn19 := D19 = \left(\frac{3}{4}\right) \cdot D20 + \left(\frac{1}{4}\right) \cdot D18 + 1;$

$eqn18 := D18 = 0;$

$$D26 = 3 + D25$$

$$D25 = \frac{1}{3} D26 + \frac{2}{3} D24 + 1$$

$$D24 = \frac{1}{3} D25 + \frac{2}{3} D23 + 1$$

$$D23 = \frac{1}{3} D24 + \frac{2}{3} D22 + 1$$

$$D22 = \frac{1}{3} D23 + \frac{2}{3} D21 + 1$$

$$D21 = \frac{9}{10} D22 + \frac{1}{10} D20 + 1$$

$$D20 = \frac{3}{4} D21 + \frac{1}{4} D19 + 1$$

$$D19 = \frac{3}{4} D20 + \frac{1}{4} D18 + 1$$

$$D18 = 0 \quad \textbf{(2.1)}$$

$durations := solve(\{eqn26, eqn25, eqn24, eqn23, eqn22, eqn21, eqn20, eqn19, eqn18\}, \{D26, D25, D24, D23, D22, D21, D20, D19, D18\});$

$$\{D18 = 0, D19 = 349, D20 = 464, D21 = 501, D22 = 504, D23 = 507, D24 = 510, D25 = 513, D26 = 516\} \quad \textbf{(2.2)}$$

$subs(\%, D25);$

$$513 \quad \textbf{(2.3)}$$

# Problem 3

Insert the trial solution

$$W_{sk}^{p} = \begin{cases} 0 & s \le k \\ -2(s-k) & s \ge k \end{cases}$$

into the difference equation. For s < k, the substitution yields

$0 = 0 + \left(\frac{1}{2}\right)\cdot 0 + 0\cdot\left(\frac{1}{2}\right)$ so the equation is satisfied by the trial particular solution.
For $s > k$ the difference equation becomes
$(2)\cdot(k-s) = 0 + \left(\frac{1}{2}\right)\cdot(2)\cdot(k-(s-1)) + \left(\frac{1}{2}\right)\cdot(2)\cdot(k-(s+1))$ and again the equation is satisfied identically.
For $s = k$, the difference equation becomes
$0 = 1 + \left(\frac{1}{2}\right)\cdot(0) + \left(\frac{1}{2}\right)\cdot(2)\cdot(k-(k+1))$. Therefore, the particular solution is

$$W^p_{sk} = \begin{cases} 0 & s \le k \\ -2(s-k) & s \ge k \end{cases} = -2\max(0, s-k).$$

# Problem 4

$$ExpectedVisits := 2\left[\frac{s}{S}\cdot\sum_{k=1}^{S-1} k\cdot(S-k) - \sum_{k=1}^{s-1} k\cdot(s-k)\right]$$

$$\left[\frac{2\,s\left(\frac{1}{6}\,S^3 - \frac{1}{6}\,S\right)}{S} - \frac{1}{3}\,s^3 + \frac{1}{3}\,s\right] \quad (4.1)$$

simplify symbolic →

$$\left[\frac{1}{3}\,s\,(S^2 - s^2)\right] \quad (4.2)$$

$$SumIntegers := \sum_{k=1}^{N} k$$

$$\frac{1}{2}\,(N+1)^2 - \frac{1}{2}\,N - \frac{1}{2} \quad (4.3)$$

simplify symbolic →

$$\frac{1}{2}\,N^2 + \frac{1}{2}\,N \quad (4.4)$$

$$SumSquares := \sum_{k=1}^{N} k^2$$

$$\frac{1}{3}\,(N+1)^3 - \frac{1}{2}\,(N+1)^2 + \frac{1}{6}\,N + \frac{1}{6} \quad (4.5)$$

simplify symbolic →

$$\frac{1}{3}\,N^3 + \frac{1}{2}\,N^2 + \frac{1}{6}\,N \quad (4.6)$$

$$SumQuadratic := \sum_{k=1}^{N} k\cdot(M-k)$$

$$\frac{1}{2} M (N+1)^2 - \frac{1}{2} M (N+1) - \frac{1}{3} (N+1)^3 + \frac{1}{2} (N+1)^2 - \frac{1}{6} N - \frac{1}{6} \quad \textbf{(4.7)}$$

$\xrightarrow{\text{simplify symbolic}}$

$$\frac{1}{2} M N^2 + \frac{1}{2} M N - \frac{1}{3} N^3 - \frac{1}{2} N^2 - \frac{1}{6} N \quad \textbf{(4.8)}$$

$$CominationPowers := M\cdot SumIntegers - SumSquares$$

$$M\left(\frac{1}{2} (N+1)^2 - \frac{1}{2} N - \frac{1}{2}\right) - \frac{1}{3} (N+1)^3 + \frac{1}{2} (N+1)^2 - \frac{1}{6} N - \frac{1}{6} \quad \textbf{(4.9)}$$

$\xrightarrow{\text{simplify symbolic}}$

$$\frac{1}{2} M N^2 + \frac{1}{2} M N - \frac{1}{3} N^3 - \frac{1}{2} N^2 - \frac{1}{6} N \quad \textbf{(4.10)}$$

# Problem 5

$$C := \frac{\left(K + \left(\frac{1}{3}\right)\cdot r\cdot S^3\cdot x\cdot(1 - x^2)\right)}{S^2\cdot x\cdot(1-x)}$$

$$\frac{K + \frac{1}{3} r S^3 x (1-x^2)}{S^2 x (1-x)} \quad \textbf{(5.1)}$$

## Part a

$$dCdx := diff(C, x);$$

$$\frac{\frac{1}{3} r S^3 (1-x^2) - \frac{2}{3} r S^3 x^2}{S^2 x (1-x)} - \frac{K + \frac{1}{3} r S^3 x (1-x^2)}{S^2 x^2 (1-x)} + \frac{K + \frac{1}{3} r S^3 x (1-x^2)}{S^2 x (1-x)^2} \quad \textbf{(5.1.1)}$$

## Part b

$$dCdS := diff(C, S);$$

$$\frac{r(1-x^2)}{1-x} - \frac{2\left(K + \frac{1}{3} r S^3 x (1-x^2)\right)}{S^3 x (1-x)} \quad \textbf{(5.2.1)}$$

## Part c

$optima := solve(\{dCdx = 0, dCdS = 0\}, \{S, x\});$

$$\left\{S = 3\,RootOf(4\,\_Z^3\,r - 3\,K),\, x = \frac{1}{3}\right\} \quad \textbf{(5.3.1)}$$

$allvalues(optima);$

$$\left\{S = \frac{3}{4}\,3^{1/3}\,4^{2/3}\left(\frac{K}{r}\right)^{1/3},\, x = \frac{1}{3}\right\},\, \left\{S = \frac{3}{4}\,3^{1/3}\,4^{2/3}\left(\frac{K}{r}\right)^{1/3}(-1)^{2/3},\, x = \frac{1}{3}\right\},\, \left\{S = -\frac{3}{4}\,3^{1/3}\,4^{2/3}\left(\frac{K}{r}\right)^{1/3}(-1)^{1/3},\, x = \frac{1}{3}\right\} \quad \textbf{(5.3.2)}$$

Inspect carefully, one solution set with is real, the other terms have complex factors $(-1)^{\frac{2}{3}}$ and $(-1)^{\frac{1}{3}}$.

We can also solve the partial derivative eqations "by hand".
Set the derivative $dCdS$ equal to 0, then clear denominators.

$r \cdot S^3 \cdot x \cdot (1 - x^2) = 2\left(K + \frac{1}{3}\, r\, S^3\, x\,(1 - x^2)\right)$

$$r\, S^3\, x\,(1 - x^2) = 2\,K + \frac{2}{3}\, r\, S^3\, x\,(1 - x^2) \quad \textbf{(5.3.3)}$$

$\xrightarrow{\text{isolate for K}}$

$$K = \frac{1}{6}\, r\, S^3\, x\,(1 - x^2) \quad \textbf{(5.3.4)}$$

Substitute this into the derivative with respect to $x$.

$subs\left(K = \frac{r \cdot S^3 \cdot x \cdot (1 - x^2)}{6},\, dCdx\right);$

$$\frac{\frac{1}{3}\, r\, S^3\,(1 - x^2) - \frac{2}{3}\, r\, S^3\, x^2}{S^2\, x\,(1 - x)} - \frac{1}{2}\,\frac{r\, S\,(1 - x^2)}{x\,(1 - x)} + \frac{1}{2}\,\frac{r\, S\,(1 - x^2)}{(1 - x)^2} \quad \textbf{(5.3.5)}$$

$\xrightarrow{\text{simplify symbolic}}$

$$-\frac{1}{6}\,\frac{(3\,x - 1)\, r\, S}{x\,(-1 + x)} \quad \textbf{(5.3.6)}$$

Now this derivative is 0 when $x = \frac{1}{3}$. Note that $x = 0$ and $x = 1$ are not realistic values of $x$, that is, the values $x = 0$ and $x = 1$ are not in the domain so the denominator is never $0$ in the domain. Likewise, $r$ and $S$ are not $0$. Then substitute $x = \frac{1}{3}$ back into the expression for $K$

$subs\left(x = \frac{1}{3},\, K = \frac{r \cdot S^3 \cdot x \cdot (1 - x^2)}{6}\right)$

$$K = \frac{4}{81} r S^3 \tag{5.3.7}$$

$\xrightarrow{\text{isolate for S}}$

$$S = RootOf(4\_Z^3 r - 81 K) \tag{5.3.8}$$

*allvalues*(%);

$$S = \frac{1}{4} 81^{1/3} 4^{2/3} \left(\frac{K}{r}\right)^{1/3}, S = \frac{1}{4} 81^{1/3} 4^{2/3} \left(\frac{K}{r}\right)^{1/3} (-1)^{2/3}, S = -\frac{1}{4} 81^{1/3} 4^{2/3} \left(\frac{K}{r}\right)^{1/3} (-1)^{1/3} \tag{5.3.9}$$