Math 489/889
Stochastic Processes and Advanced Mathematical Finance
Homework 4

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Due Sept 27, 2010

1. Consider a stock whose price today is \$50. Suppose that over the next year, the stock price can either go up by 6%, or down by 3%, so the stock price at the end of the year is either \$53 or \$48.50. The continuously compounded interest rate on a \$1 bond is 4%. If there also exists a call option on the stock with an exercise price of \$50, then what is the price of the call option? Also, what is the replicating portfolio?

   Solution: Solve

$$\phi \cdot 53 + \psi \cdot e^{0.04 \cdot 1} = 3$$
$$\phi \cdot 48.50 + \psi \cdot e^{0.04 \cdot 1} = 0$$

   to obtain $\phi = 0.6666666667$ and $\psi = -31.06552521$, so that $V = \phi \cdot 50 + \psi = 0.6666666667 \cdot 50 - 31.06552521 = 2.26780813$.

2. A stock price is currently \$50. It is known that at the end of 6 months, it will either be \$60 or \$42. The risk-free rate of interest with continuous compounding on a \$1 bond is 10% per year. Calculate the value of a 6-month European call option on the stock with strike price \$48 and find the replicating portfolio.

Solution: Solve

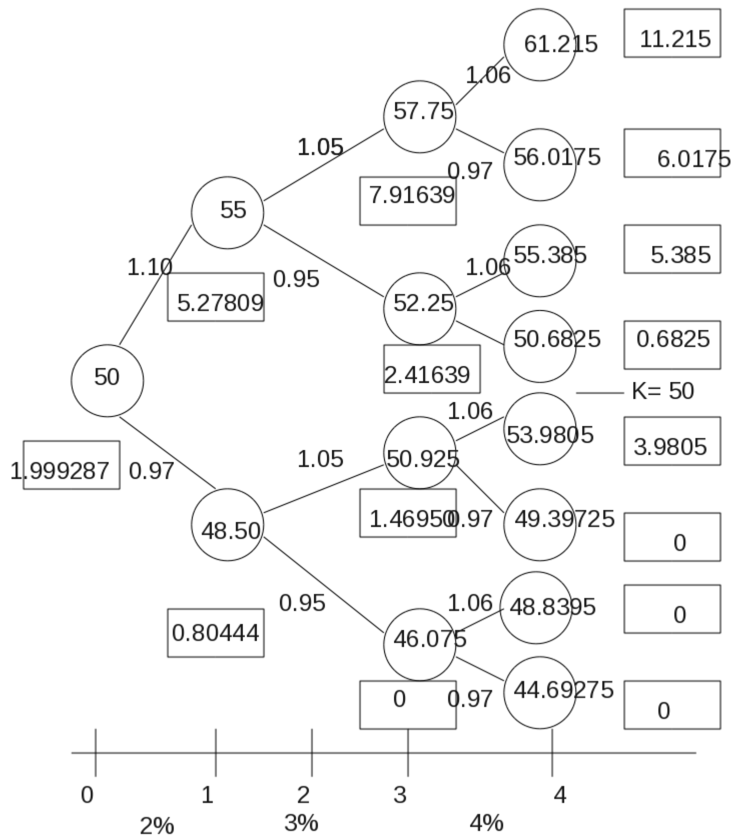
$$\phi \cdot 60 + \psi \cdot e^{0.05} = 12$$
$$\phi \cdot 42 + \psi \cdot e^{0.05} = 0$$

to obtain $\phi = 0.6666666667$ and $\psi = -26.63442390$ so that $V = \phi \cdot 50 + \psi = 6.69890944$.

3. Consider a three-time-stage example. The first time interval is a month, then the second time interval is two months, finally, the third time interval is a month again. A stock starts at 50. In the first interval, the stock can go up by 10% or down by 3%, in the second interval the stock can go up by 5% or down by 5%, finally in the third time interval, the stock can go up by 6% or down by 3%. The continuously compounded interest rate on a \$1 bond is 2% in the fist period, 3% in the second period, and 4% in the third period. Find the price of a call option with exercise price 50, with exercise date at the end of the 4 months. Also, find the replicating portfolio at each node.

   Solution: The values of the binomial tree of outcomes are:

Node	Stock	Option
$V_{3,7}$	61.215	11.215
$V_{3,6}$	56.0175	6.0175
$V_{3,5}$	55.385	5.385
$V_{3,4}$	50.6825	0.6825
$V_{3,3}$	53.9805	3.9805
$V_{3,2}$	49.39725	0
$V_{3,1}$	48.8395	0
$V_{3,0}$	44.69275	0
$V_{2,3}$	57.75	7.91639
$V_{2,2}$	52.25	2.41639
$V_{2,1}$	50.925	1.46960
$V_{2,0}$	46.075	0
$V_{1,1}$	55.00	5.27809
$V_{1,0}$	48.50	0.80444
$V_{0,0}$	50	1.99929



4. A *long strangle option* pays $\max(K_1 - S, 0, S - K_2)$ if it expires when the underlying stock value is S. The parameters K_1 and K_2 are the lower strike price and the upper strike price, and $K_1 < K_2$. A stock currently has price \$100 and goes up or down by 20% in each time period. What is the value of such a long strangle option with lower strike 90 and upper strike 110 at expiration 2 time units in the future? Assume a simple interest rate of 10% in each time period.

Solution:

Node	Stock	Option
$V_{2,3}$	144	34
$V_{2,2}$	96	0
$V_{2,1}$	96	0
$V_{2,0}$	64	26
$V_{1,1}$	120	23.1818
$V_{1,0}$	80	5.9091
$V_{0,0}$	100	16.9628

5. Your friend, the financial analyst comes to you, the mathematical economist, with a proposal: "The single period binomial pricing is all right as far as it goes, but it certainly is simplistic. Why not modify it slightly to make it a little more realistic? Specifically, assume the stock can take *three* values at time T, say it goes up by a factor U with probability p_U, it goes down by a factor D with probability p_D, where $D < 1 < U$ and the stock stays somewhere in between, changing by a factor M with probability p_M where $D < M < U$ and $p_D + p_M + p_U = 1$." The market contains only this stock, a bond with a continuously compounded risk-free rate r and an option on the stock with payoff function $f(S_T)$. Make a mathematical model based on your friend's suggestion and provide a critique of the model based on the classical applied mathematics criteria of existence of solutions to the model and uniqueness of solutions to the model.

   Solution: This would lead to the system of three equations in two unknowns:

$$\phi \cdot SU + \psi e^{rt} = f(SU)$$
$$\phi \cdot SM + \psi e^{rt} = f(SM)$$
$$\phi \cdot SD + \psi e^{rt} = f(SD)$$

   Since we assume that $U > M > D$, the system has no solution except for the very special circumstance (for instance) that

$$-\frac{(f(SM) - f(SU))\, D}{U - M} + \frac{Uf(SM) - Mf(SU)}{U - M} = f(SD).$$

   This is a flawed model as it stands, since generally there is no solution to three independent equations in two unknowns. The solution does not exist. We say the problem is ill-posed.