Math 489/889

Stochastic Processes and

Advanced Mathematical Finance

Homework 10

Steve Dunbar

Due Wednesday, December 8, 2010

Problem 1

Simulate the solution of the stochastic differential equation dX(t) = X(t)dt + 2*X(t)* dW on the interval [0, 1] with initial condition X (0) = 1 and step size dt = 1/10. Put your calculations in a table as done in the lesson. Include a record of your coin flips.

Note that below I using a freshly generated sum of 10 coin flips in each time step of 1/10. I am using

$W_{100}(t) = \frac{1}{10} \cdot W(100\,t)$ as my approximation to Brownian Motion, *not the approximation* $W_{10}(t) = \frac{1}{\sqrt{10}} \cdot W(10\,t)$ that I recommended that you use.

with(*Statistics*);

[*AbsoluteDeviation*, *AgglomeratedPlot*, *AreaChart*, *BarChart*, *Bootstrap*, *BoxPlot*, *BubblePlot*, *CDF*, *CGF*, *CentralMoment*, *CharacteristicFunction*, *ChiSquareGoodnessOfFitTest*, *ChiSquareIndependenceTest*, *ChiSquareSuitableModelTest*, *ColumnGraph*, *Correlation*, *CorrelationMatrix*, *Count*, *CountMissing*, *Covariance*, *CovarianceMatrix*, *Cumulant*, *CumulantGeneratingFunction*, *CumulativeDistributionFunction*, *CumulativeProduct*, *CumulativeSum*, *CumulativeSumChart*, *DataSummary*, *Decile*, *DensityPlot*, *Discretize*, *Distribution*, *ErrorPlot*, *EvaluateToFloat*, *Excise*, *ExpectedValue*, *ExponentialFit*, *ExponentialSmoothing*, *FailureRate*, *FisherInformation*, *Fit*, *FivePointSummary*, *FrequencyPlot*, *FrequencyTable*, *GeometricMean*, **(1.1)**

HarmonicMean, HazardRate, Histogram, Information, InteractiveDataAnalysis, InterquartileRange, InverseSurvivalFunction, Join, KernelDensity, KernelDensityPlot, KernelDensitySample, Kurtosis, Likelihood, LikelihoodRatioStatistic, LineChart, LinearFilter, LinearFit, LogLikelihood, LogarithmicFit, MGF, MLE, MakeProcedure, MaximumLikelihoodEstimate, Mean, MeanDeviation, Median, MedianDeviation, MillsRatio, Mode, Moment, MomentGeneratingFunction, MovingAverage, MovingMedian, MovingStatistic, NonlinearFit, NormalPlot, OneSampleChiSquareTest, OneSampleTTest, OneSampleZTest, OneWayANOVA, OrderByRank, OrderStatistic, PDF, Percentile, PieChart, PointPlot, PolynomialFit, PowerFit, Probability, ProbabilityDensityFunction, ProbabilityFunction, ProbabilityPlot, ProfileLikelihood, ProfileLogLikelihood, QuadraticMean, Quantile, QuantilePlot, Quartile, RandomVariable, Range, Rank, Remove, RemoveInRange, RemoveNonNumeric, Sample, ScatterPlot, Score, Select, SelectInRange, SelectNonNumeric, ShapiroWilkWTest, Shuffle, Skewness, Sort, StandardDeviation, StandardError, StandardizedMoment, SunflowerPlot, Support, SurfacePlot, SurvivalFunction, SymmetryPlot, Tally, TallyInto, Trim, TrimmedMean, TwoSampleFTest, TwoSamplePairedTTest, TwoSampleTTest, TwoSampleZTest, Variance, Variation, WeightedMovingAverage, Winsorize, WinsorizedMean]

$X := 2 \cdot RandomVariable\left(Binomial\left(10, \frac{1}{2}\right)\right) - 10;$

$$2\,\_R - 10 \quad (1.2)$$

$Sample(X, 1)[1]$

$$2. \quad (1.3)$$

Spreadsheet(1)

	A	B	C	D	E	F	G	H
1	*Step*	"t_j"	"X_j"	"X_j dt"	"dW"	"2*X_j*dW"	"X_j dt + 2X_j dW"	"X_{j+1}"
2	0	0	1	$\frac{1}{10}$	0.	0.	$\frac{1}{10}$	$\frac{11}{10}$
3	1	$\frac{1}{10}$	$\frac{11}{10}$	$\frac{11}{100}$	0.2000000\ 000	0.4400000000	0.5500000000	1.650000000
4	2	$\frac{1}{5}$	1.650000000	0.1650000000	-0.4	-1.320000000	-1.155000000	0.495000000
5	3	$\frac{3}{10}$	0.495000000	0.04950000000	0.	0.	0.04950000000	0.5445000000
6	4	$\frac{2}{5}$	0.5445000000	0.05445000000	0.4000000\ 000	0.4356000000	0.4900500000	1.034550000
7	5	$\frac{1}{2}$	1.034550000	0.1034550000	0.2000000\ 000	0.4138200000	0.5172750000	1.551825000
8	6	$\frac{3}{5}$	1.551825000	0.1551825000	0.6000000\ 000	1.862190000	2.017372500	3.569197500
9	7	$\frac{7}{10}$	3.569197500	0.3569197500	0.2000000\ 000	1.427679000	1.784598750	5.353796250
10	8	$\frac{4}{5}$	5.353796250	0.5353796250	-0.6	-6.424555500	-5.889175875	-0.535379625
11	9	$\frac{9}{10}$	-0.535379625	-0.05353	0.4000000\ 000	-0.4283037000	-0.4818416625	-1.017221288
12	10	1	-1.017221288	-0.1017221288	0.4000000\	-0.8137770304	-0.9154991592	-1.932720447
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Problem 2

Find the solution of the stochastic differential equation

dX(t) = X(t) dt + 2 X(t) dW

Solution: Guess a solution of the form

$$X(t) = e^{a\,t + b\,W(t)}$$

where a and b are parameters to be determined. Then by Ito's formula

dX = (a + b^2/2) X dt + b X dW

so matching coefficients $a + \frac{b^2}{2} = 1$ and b = 2, so

a = -1. The solution is $X(t) = e^{-t + 2\,W(t)}$.

Problem 3

Find the mode (the value of the independent variable with the highest probability) of the lognormal probability density function. (Use parameters μ and σ.)

$restart;$

$with(Statistics):$

$X := RandomVariable(LogNormal(\mu, \sigma)):$

Note very carefully that Maple uses different parameters than I do! I define the lognormal in terms of the parameters μ and σ^2, but Maple uses the parameters μ and σ to define the same distribution.
So be careful to get the distribution you really want!

$f := PDF(X, x)$

$$\begin{cases} 0 & x < 0 \\ \frac{1}{2} \frac{\sqrt{2}\, e^{-\frac{1}{2} \frac{(\ln(x) - \mu)^2}{\sigma^2}}}{x \sigma \sqrt{\pi}} & otherwise \end{cases} \tag{3.1}$$

$df := diff(f, x);$

$$\begin{cases} 0 & x < 0 \\ undefined & x = 0 \\ -\frac{1}{2} \frac{\sqrt{2}\, e^{-\frac{1}{2} \frac{(\ln(x) - \mu)^2}{\sigma^2}}}{x^2 \sigma \sqrt{\pi}} - \frac{1}{2} \frac{\sqrt{2}\, (\ln(x) - \mu)\, e^{-\frac{1}{2} \frac{(\ln(x) - \mu)^2}{\sigma^2}}}{x^2 \sigma^3 \sqrt{\pi}} & 0 < x \end{cases} \tag{3.2}$$

$solve(df = 0, x)$ assuming $x > 0$;

$$e^{\mu - \sigma^2} \tag{3.3}$$

Problem 4

Suppose you buy a stock (or more properly an index fund) whose value $S(t)$ is described by the stochastic differential equation
$dS = 0\, S\, dt + \sigma\, S\, dW$
corresponding to a zero compounded growth and pure market fluctuations proportional to the stock value. What are your chances to double your money?

$$gamblersruinformula := \frac{\left(1 - A^{1 - \frac{(2 \cdot \mu - \sigma^2)}{\sigma^2}}\right)}{\left(B^{1 - \frac{(2 \cdot \mu - \sigma^2)}{\sigma^2}} - A^{1 - \frac{(2 \cdot \mu - \sigma^2)}{\sigma^2}}\right)}$$

$$\frac{1 - A^{1 - \frac{2\mu - \sigma^2}{\sigma^2}}}{B^{1 - \frac{2\mu - \sigma^2}{\sigma^2}} - A^{1 - \frac{2\mu - \sigma^2}{\sigma^2}}} \tag{4.1}$$

$\mu := -\frac{\sigma^2}{2};$

$$-\frac{1}{2}\sigma^2$$

$A := 0;$

$$0$$

$B := 2;$

$$2$$

gamblersruinformula;

$$\frac{1}{8} \tag{4.5}$$

Suppose you buy a stock (or more properly an index fund) whose value $S(t)$ is described by the stochastic differential equation

$$dS = \frac{\sigma^2}{2} \cdot S\,dt + \sigma \cdot dW$$

corresponding to a non-zero compounded growth and pure market fluctuations proportional to the stock value and the two parameter values are coincidentally connected in their value. What are your chances to double your money?

$\mu := 0;$
$A := 0;$
$B := 2;$

$$0$$

$$0$$

$$2 \tag{4.6}$$

gamblersruinformula;

$$\frac{1}{4} \tag{4.7}$$