1. Let $W(t)$ be standard Brownian motion.

(a) Find the probability that $0 < W(1) < 1$.

(b) Find the probability that $0 < W(1) < 1$ and $1 < W(2) < 3$.

(c) Find the probability that $0 < W(1) < 1$ and $1 < W(2) < 3$ and $0 < W(3) < 1/2$.

Solution:

(a) $P[0 < W(1) < 1] = \Phi(1) - \Phi(0) \approx 0.3413$

(b)

$$P[0 < W(1) < 1, 1 < W(2) < 3] = \int_1^3 \int_0^1 \frac{\exp(-x_1^2/(2 \cdot 1)) \cdot \exp(-(x_2 - x_1)^2/(2 \cdot 1))}{\sqrt{2\pi \cdot 1}} \, dx_1 \, dx_2$$

$$\approx 0.100$$
(c) \[
\mathbb{P}[0 < W(1) < 1, 1 < W(2) < 3, 0 < W(3) < 1/2] = \\
\int_0^{1/2} \int_1^3 \int_0^1 \exp\left(-\frac{x_1^2}{2 \cdot 1}\right) \cdot \exp\left(-\frac{(x_2 - x_1)^2}{2 \cdot 1}\right) \cdot \\
\exp\left(-\frac{(x_3 - x_2)^2}{2 \cdot 1}\right) \frac{dx_1}{\sqrt{2\pi \cdot 1}} \cdot \frac{dx_2}{\sqrt{2\pi \cdot 1}} \cdot \frac{dx_3}{\sqrt{2\pi \cdot 1}} = 0.00816
\]

2. Let \( W(t) \) be standard Brownian motion.

(a) Evaluate the probability that \( W(5) \leq 3 \) given that \( W(1) = 1 \).

(b) Find the number \( c \) such that \( \mathbb{P}[W(9) > c | W(1) = 1] = 0.10 \).

Solution:

(a) Since \( W(5) - W(1) \sim N(0, 4) \), the required probability is
\[
\mathbb{P}[W(5) > 3 | W(1) = 1] = \mathbb{P}[W(5) - W(1) > 3 - 1] \\
= \mathbb{P}\left[\frac{(W(5) - W(1))/2}{\sqrt{2}} > 1\right] \\
= 0.1586552539
\]

(b) Since \( W(9) - W(1) \sim N(0, 8) \), the required value can be deduced from
\[
\mathbb{P}[W(9) > c | W(1) = 1] = \mathbb{P}[W(9) - W(1) > c - 1] \\
= \mathbb{P}\left[\frac{(W(9) - W(1))/\sqrt{2}}{2} > \frac{(c - 1)/\sqrt{2}}{2}\right] \\
= 0.10
\]

Then \( (c - 1)/2\sqrt{2} = 1.281551566 \) and \( c = 4.624775211 \).

3. Use your September 21, 2009 record of a 100-flip coin flip sequence. Scoring \( Y_i = +1 \) for each Head and \( Y_i = -1 \) for each Tail on each flip, keep track of the accumulated sum \( T_n = \sum_{i=1}^n Y_i \) for \( n = 1, \ldots, 100 \) representing the net fortune at any time. Plot the resulting \( T_n \) versus \( n \) on the interval \([0, 100]\). Finally, using \( N = 10 \) plot the rescaled approximation \( W_{10}(t) = (1/\sqrt{10})S(10t) \) on the interval \([0, 10]\) on the same graph.
4. Let $Z$ be a single normally distributed random variable, with mean 0 and variance 1, i.e. $Z \sim N(0,1)$. Then consider the continuous time stochastic process $X(t) = \sqrt{t}Z$.

(a) Using a normal random variable generator (Excel, Maple, Mathematica, MATLAB, R etc., all have one and probably the TI-89 or equivalent has one too), find sample values of $X(1)$, $X(2)$, $X(4)$ and $X(9)$.

(b) Explain why the distribution of $X(t)$ is normal with mean 0 with variance $t$.

(c) Is $X(t)$ a Brownian motion? Explain why or why not.

Solution: No, $X(t)$ is not Brownian motion for two reasons in spite of the fact that $\sqrt{t}Z \sim N(0,t)$ (which follows from being a scalar multiple by $\sqrt{t}$ of the $N(0,1)$ random variable $Z$.)

First, for $t_1 < t_2 < t_3 < t_4$, $X(t_2) - X(t_1) = (\sqrt{t_2} - \sqrt{t_1})Z$ is not independent of $X(t_4) - X(t_3) = (\sqrt{t_4} - \sqrt{t_3})Z$ since both are multiples of the same sample value $Z$ drawn from the $N(0,1)$ population.

Second, the distribution of $(\sqrt{t_2} - \sqrt{t_1})Z$ is normal with variance $(\sqrt{t_2} - \sqrt{t_1})^2 \neq t_2 - t_1$.

Note nevertheless that $X(0) = 0$ and $X(t)$ is continuous at $t = 0$.

5. What is the distribution of $W(s) + W(t)$, for $0 \leq s \leq t$? (Hint: Note that $W(s)$ and $W(t)$ are not independent. But by using the “add-in-subtract-out trick” you can write $W(s) + W(t)$ as a sum of 3 random variables. This problem requires almost no calculation, but it does require insight into how to rewrite the given expression in terms of increments.)

Solution:

\[
W(s) + W(t) = (W(s) - W(0)) + W(s) + (W(t) - W(s))
\]

\[
= 2(W(s) - W(0)) + (W(t) - W(s))
\]
and \((W(s) - W(0))\) is independent from \((W(t) - W(s))\), so

\[
W(s) + W(t) = 2(W(s) - W(0)) + (W(t) - W(s)) \sim N(0, 2^2 s + (t - s)) = N(0, 3s + t).
\]

6. Show that the probability density function

\[
p(t; x, y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - y)^2}{2t}\right)
\]

satisfies the partial differential equation for heat flow (the heat equation)

\[
\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}.
\]

Solution:

\[
\frac{\partial p}{\partial t} = -\frac{1}{\sqrt{2\pi t^{3/2}}} \exp\left(-\frac{(x - y)^2}{2t}\right) + \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - y)^2}{2t}\right) \frac{(x - y)^2}{2t^2}
\]

\[
\frac{\partial p}{\partial x} = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - y)^2}{2t}\right) \frac{(x - y)}{t}
\]

\[
\frac{\partial^2 p}{\partial x^2} = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - y)^2}{2t}\right) \frac{(x - y)^2}{t^2} + \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{(x - y)^2}{2t}\right) \frac{1}{t}
\]

Now after a little combining, the first summand in \(\frac{\partial p}{\partial t}\) matches the second summand in \(\frac{1}{2} \frac{\partial^2 p}{\partial x^2}\) and the second summand in \(\frac{\partial p}{\partial x}\) matches the first summand in \(\frac{1}{2} \frac{\partial^2 p}{\partial x^2}\).

7. For two random variables \(X\) and \(Y\), statisticians call

\[
\text{Cov}[X, Y] = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]
\]

the covariance of \(X\) and \(Y\). If \(X\) and \(Y\) are independent, then \(\text{Cov}[X, Y] = 0\). A positive value of \(\text{Cov}[X, Y]\) indicates that \(Y\) tends to values greater than its mean if \(X\) is greater than its mean, while a negative
value indicates that $Y$ tends to values below its mean when $X$ is above its mean. Thus, $\text{Cov} \ [X, Y]$ is an indication of the mutual dependence of $X$ and $Y$. If $W(t)$ is the standard Wiener process, show that

$$\text{Cov} \ [W(s), W(t)] = \mathbb{E}[W(s)W(t)] = \min(t, s).$$

Assuming that a stock price changes according to Brownian motion, interpret this for a stock at $t$ and $t + 1$.

**Solution:**

$$\text{Cov} \ [W(s), W(t)] = \mathbb{E}[(W(s) - \mathbb{E}[W(s)]) \cdot (W(t) - \mathbb{E}[W(t)])]$$

$$= \mathbb{E}[W(s) \cdot W(t)].$$

Without loss of generality, assume $s < t$. Then

$$\mathbb{E}[W(s) \cdot W(t)] = \mathbb{E}[W(s) \cdot (W(t) - W(s) + W(s))]$$

$$= \mathbb{E}[W(s) \cdot (W(t) - W(s))] + \mathbb{E}[W(s)W(s)]$$

$$= 0 + \text{Var}[W(s)] = s.$$

The process is exactly analogous if $t < s$, so

$$\text{Cov} \ [W(s), W(t)] = \mathbb{E}[W(s)W(t)] = \min(t, s).$$

The interpretation is that a stock price change is above its mean value on day $t$ is likely to also be above its mean stock price change on day $t + 1$.

8. **Required for Mathematics Graduate Students, Extra Credit for anyone else** Let $W(t)$ be a standard Brownian motion. Let $\epsilon$ be a positive value.

(a) Show that

$$\mathbb{P} \left[ \frac{|W(t)|}{t} > \epsilon \right] = 2(1 - \Phi(\epsilon \sqrt{t}))$$

where $\Phi(\cdot)$ is the cdf for a $N(0, 1)$ random variable.
(b) How does $\Pr \left[ \frac{|W(t)|}{\sqrt{t}} > \epsilon \right]$ behave when $t \to \infty$? How does this behave when $t \to 0$? Both of these questions should be answered by finding the asymptotic rate of convergence, that is if $f(t) = 2(1 - \Phi(\epsilon \sqrt{t}))$, find a function $g(t)$ such that $\lim(f(t)/g(t))$ exists, or equivalently showing that there are constants $C_1$ and $C_2$ such that $C_1 g(t) \leq f(t) \leq C_2 g(t)$ as the independent variable $t$ approaches its limit. A suggestion for an appropriate function $g(t)$ is to use the leading term of a series expansion.

(c) Interpret these two asymptotic statements geometrically.

Solution: Rewrite the probability as $\Pr \left[ \frac{|W(t)|}{\sqrt{t}} > \epsilon \sqrt{t} \right]$ and then $W(t)/\sqrt{t} \sim N(0,1)$ so the result in terms of $\Phi$ follows immediately.

To understand the behavior as $t \to \infty$, we can expand this function in a series expansion with the Maple `asympt` command and obtain

$$\left( \frac{\sqrt{2} \sqrt{t^{-1}}}{\sqrt{\pi} \epsilon} - \frac{\sqrt{2} (t^{-1})^{3/2}}{e^3 \sqrt{\pi}} + 3 \frac{\sqrt{2} (t^{-1})^{5/2}}{e^5 \sqrt{\pi}} - 15 \frac{\sqrt{2} (t^{-1})^{7/2}}{e^7 \sqrt{\pi}} \right) \frac{1}{\sqrt{\exp(\epsilon^2 t)}}.$$  

Then we can show

$$2(1 - \Phi(\epsilon \sqrt{t})) \sim \frac{\sqrt{2}}{\epsilon \sqrt{\pi} \cdot t \cdot \exp(\epsilon^2 t)}.$$  

To understand the behavior as $t \to 0$, we can expand this function in a series expansion with the Maple `series` command and obtain

$$1 - \frac{\sqrt{2} \epsilon \sqrt{t}}{\sqrt{\pi}} + 1/6 \frac{e^3 \sqrt{2} t^{-3/2}}{\sqrt{\pi}} - 1/40 \frac{e^5 \sqrt{2} t^{5/2}}{\sqrt{\pi}} + \frac{1}{336} \frac{e^7 \sqrt{2} t^{7/2}}{\sqrt{\pi}} + O(t^4).$$  

Then we can show

$$2(1 - \Phi(\epsilon \sqrt{t})) \sim 1 - \epsilon \sqrt{\frac{2t}{\pi}}.$$  

The limit at infinity says that the probability that the absolute value of Brownian Motion exceeds any linear function, no matter how shallow the slope, with probability going to zero at $t$ goes to infinity. That
is, for any $\epsilon > 0$, the probability $|W(t)|$ is less than the function $\epsilon t$ as $t$ approaches $\infty$ goes to 1. In fact, it says more: It says that with probability going to 1 at an exponential rate $\exp(-\epsilon^2 t)/\sqrt{t}$, the ratio $W(t)/(\epsilon t)$ goes to 0. In summary, with probability approaching 1 rapidly in the independent variable, Brownian motion is sublinear as the independent variable goes to infinity.

The limit at zero says that the probability that the absolute value of Brownian Motion exceeds any linear value, no matter how steep, with probability going to one at $t$ goes to infinity. That is, for any $\epsilon > 0$, the probability $|W(t)|$ is less than the function $\epsilon t$ as $t$ approaches 0 goes to 1. In fact, it says more: It says that with probability going to 1 at a rate $\epsilon \sqrt{\frac{2t}{\pi}}$, the ratio $|W(t)|/(\epsilon t) > 1$. In summary, with probability approaching 1 in the independent variable, Brownian motion is superlinear as the independent variable goes to zero.

This is a very weak form of the kind of limit statements found in the “law of the iterated logarithm.”
