1. Let $W(t)$ be standard Brownian motion.

   (a) Find the probability that $0 < W(1) < 1$.
   (b) Find the probability that $0 < W(1) < 1$ and $1 < W(2) < 3$.
   (c) Find the probability that $0 < W(1) < 1$ and $1 < W(2) < 3$ and $0 < W(3) < 1/2$.

2. Let $W(t)$ be standard Brownian motion.

   (a) Evaluate the probability that $W(5) \leq 3$ given that $W(1) = 1$.
   (b) Find the number $c$ such that $\Pr[W(9) > c | W(1) = 1] = 0.10$.

3. Use your September 21, 2009 record of a 100-flip coin flip sequence. Scoring $Y_i = +1$ for each Head and $Y_i = -1$ for each Tail on each flip, keep track of the accumulated sum $T_n = \sum_{i=1}^{n} Y_i$ for $n = 1, \ldots, 100$ representing the net fortune at any time. Plot the resulting $T_n$ versus $n$ on the interval $[0, 100]$. Finally, using $N = 10$ plot the rescaled approximation $W_{10}(t) = (1/\sqrt{10})S(10t)$ on the interval $[0, 10]$ on the same graph.

4. Let $Z$ be a single normally distributed random variable, with mean 0 and variance 1, i.e. $Z \sim N(0, 1)$. Then consider the continuous time stochastic process $X(t) = \sqrt{t}Z$. 

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(a) Using a normal random variable generator (Excel, Maple, Mathematica, MATLAB, R etc., all have one and probably the TI-89 or equivalent has one too), find sample values of \( X(1), X(2), X(4) \) and \( X(9) \).

(b) Explain why the distribution of \( X(t) \) is normal with mean 0 with variance \( t \).

(c) Is \( X(t) \) a Brownian motion? Explain why or why not.

5. What is the distribution of \( W(s) + W(t) \), for \( 0 \leq s \leq t \)? (Hint: Note that \( W(s) \) and \( W(t) \) are not independent. But by using the “add-in-subtract-out trick” you can write \( W(s) + W(t) \) as a sum of 3 independent random variables. This problem requires almost no calculation, but it does require insight into how to rewrite the given expression in terms of increments.)

6. Show that the probability density function

\[
p(t; x, y) = \frac{1}{\sqrt{2\pi t}} \exp(- (x - y)^2/(2t))
\]

satisfies the partial differential equation for heat flow (the heat equation)

\[
\frac{\partial p}{\partial t} = \frac{1}{2} \frac{\partial^2 p}{\partial x^2}.
\]

7. For two random variables \( X \) and \( Y \), statisticians call

\[
\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]
\]

the covariance of \( X \) and \( Y \). If \( X \) and \( Y \) are independent, then \( \text{Cov} X, Y = 0 \). A positive value of \( \text{Cov}(X, Y) \) indicates that \( Y \) tends to values greater than its mean if \( X \) is greater than its mean, while a negative value indicates that \( Y \) tends to values below its mean when \( X \) is above its mean. Thus, \( \text{Cov} X, Y \) is an indication of the mutual dependence of \( X \) and \( Y \). If \( W(t) \) is the standard Wiener process, show that

\[
\text{Cov}(W(s), W(t)) = \mathbb{E}[W(s)W(t)] = \min(t, s).
\]

Assuming that a stock price changes according to Brownian motion, interpret this for a stock at \( t \) and \( t + 1 \).
8. **Required for Mathematics Graduate Students, Extra Credit for anyone else** Let $W(t)$ be a standard Brownian motion. Let $\epsilon$ be a positive value.

(a) Show that

$$\mathbb{P}\left(\frac{|W(t)|}{t} > \epsilon\right) = 2(1 - \Phi(\epsilon\sqrt{t}))$$

where $\Phi(\cdot)$ is the cdf for a $N(0, 1)$ random variable.

(b) How does $\mathbb{P}\left(\frac{|W(t)|}{t} > \epsilon\right)$ behave when $t \to \infty$? How does this behave when $t \to 0$? Both of these questions should be answered by finding the asymptotic rate of convergence, that is if $f(t) = 2(1 - \Phi(\epsilon\sqrt{t}))$, find a function $g(t)$ such that $\lim(f(t)/g(t))$ exists, or equivalently showing that there are constants $C_1$ and $C_2$ such that $C_1g(t) \leq f(t) \leq C_2g(t)$ as the independent variable $t$ approaches its limit. A suggestion for an appropriate function $g(t)$ is to use the leading term of a series expansion.

(c) Interpret these two asymptotic statements geometrically.