1. Find the mode (the value of the independent variable with the highest probability) of the lognormal probability density function. (Use parameters $\mu$ and $\sigma$.)

Solution: The probability density function is

$$\frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right)$$

so the derivative with respect to the independent variable $x$ is:

$$\frac{-(\ln(x) - \mu)}{\sqrt{2\pi}\sigma^3x^2} \exp\left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right) - \frac{1}{\sqrt{2\pi}\sigma x^2} \exp\left(-\frac{(\ln(x) - \mu)^2}{2\sigma^2}\right).$$

Setting this equal to zero and solving, $x_{\text{mode}} = \exp(\mu - \sigma^2)$. It is easily verified that this is a maximum.

2. (a) Suppose you buy a stock (or more properly an index fund) whose value $S(t)$ is described by the stochastic differential equation $dS = 0 \cdot S \, dt + \sigma \cdot S \, dW$ corresponding to a zero compounded growth and pure market fluctuations proportional to the stock value. What are your chances to double your money?

(b) Suppose you buy a stock (or more properly an index fund) whose value $S(t)$ is described by the stochastic differential equation $dS =
\[ \frac{\sigma^2}{2} \cdot S \, dt + \sigma \cdot S \, dW \] corresponding to a nonzero compounded growth and pure market fluctuations proportional to the stock value and the two parameter values are coincidentally connected in their value. What are your chances to double your money?

**Solution:**

(a) From the section on Itô’s Formula, the solution of the SDE \( dS = \sigma S \, dW \) is Dolean’s exponential \( S(t) = \exp(-\sigma^2 t/2 + \sigma W(t)) \). That is, the solution is a Geometric Brownian Motion with parameters \(-\sigma^2/2\) and \(\sigma\).

Now use the “gambler’s ruin” formula for Geometric Brownian Motion from the section on Geometric Brownian Motion:

\[
P\left[ \frac{z_0 \exp(rT_{A,B} + \sigma W(T_{A,B}))}{z_0} = B \right] = \frac{1 - A^{1-(2r-\sigma^2)/\sigma^2}}{B^{1-(2r-\sigma^2)/\sigma^2} - A^{1-(2r-\sigma^2)/\sigma^2}}
\]

with \(A = 0\) and \(B = 2\) to obtain

\[
P\left[ \exp(-\sigma^2 T_{0,2}/2 + \sigma W(T_{0,2})) = 2 \right] = \frac{1}{2^{1-(\sigma^2-\sigma^2)/\sigma^2}} = \frac{1}{2^{1/2}} = \frac{1}{4}\]

(b) From the section on Itô’s Formula, the solution of the SDE \( dS = (\sigma^2/2)S \, dt + \sigma S \, dW \) is \( S(t) = \exp(\sigma W(t)) \). That is, the solution is a Geometric Brownian Motion with parameters 0 and \(\sigma\).

Use the “gambler’s ruin” formula for Geometric Brownian Motion again with \(A = 0\) and \(B = 2\) to obtain

\[
P\left[ \exp(\sigma W(T_{0,2})) = 2 \right] = \frac{1}{2^{1-(0-\sigma^2)/\sigma^2}} = \frac{1}{2^1} = \frac{1}{4},
\]

3. A call option on a stock is said to be *in the money* if the value of the stock is higher than the strike price, so that selling or exercising the call option would result in a profit (ignoring transaction costs.) Consider a stock (or more properly an index fund) whose value \( S(t) \) is described by the stochastic differential equation \( dS = r \cdot S \, dt + \sigma \cdot S \, dW \) corresponding to a nonzero compounded growth and pure market fluctuations proportional to the stock value and has a current market price of \( z_0 \)? What is the probability that a call option is in the money based on a strike price \( K = 1.25z_0 \) at expiration \( T \) time units later?
Evaluate for $K = 1.25z_0$, $T = 1/2$, $r = 0.04$, and $\sigma = 0.10$ and also for $K = z_0$ with the same parameters.

*Solution:* We solve the SDE to see that $S(t) = z_0 \exp((r - \sigma^2/2)t + \sigma W(t))$ is Geometric Brownian Motion with parameters $\mu = r - \sigma^2/2$ and $\sigma$. We seek

$$P[S(T) > K] = P[z_0 \exp((r - \sigma^2/2)T + \sigma W(T)) > 1.25z_0]$$

$$= P[\exp((r - \sigma^2/2)T + \sigma W(T)) > 1.25]$$

$$= P[(r - \sigma^2/2)T + \sigma W(T) > \log(1.25)]$$

$$= P[W(T) > (\log(1.25) - (r - \sigma^2/2)T)/\sigma]$$

$$= P[Z > (\log(1.25) - (r - \sigma^2/2)T)/(\sigma \sqrt{T})]$$

$$= 1 - \Phi((\log(1.25) - (r - \sigma^2/2)T)/(\sigma \sqrt{T}))$$

$$= 1 - \Phi((\log(1.25) - (0.04 - (0.10)^2/2)(1/2))/(0.10 \sqrt{1/2}))$$

$$= 0.0018173522$$

For $K = z_0$, the numerical calculation is $1 - \Phi((\log(1) - (0.04 - (0.10)^2/2)(1/2))/(0.10 \sqrt{1/2})) = 0.5977344689$.

4. What is the price of a European call option on a non-dividend-paying stock when the stock price is $52$, the strike price is $50$, the risk-free interest rate is 12% per annum (compounded continuously), the volatility is 30% per annum, and the time to maturity is 3 months?

*Solution:* This can be evaluated in Maple with the commands, `with(finance);` and `evalf(blackscholes(52, 50, .12, 1/4, .30))` to obtain 5.057386760.
In more detail,

\[
d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}
= \frac{\log(\frac{52}{50}) + (0.12 + 0.30^2/2)(1/4)}{0.30 \sqrt{1/4}}
= 0.5364714210
\]

\[
d_2 = \frac{\log(S/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}
= \frac{\log(\frac{52}{50}) + (0.12 - 0.30^2/2)(1/4)}{0.30 \sqrt{1/4}}
= 0.3864714210
\]

Then \(\Phi(0.5364714210) = 0.704183608830784014\) and \(\Phi(0.3864714210) = 0.650426217823152353\). Finally,

\[
S\Phi\left(\frac{\log(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}\right)
- Ke^{-r(T-t)}\Phi\left(\frac{\log(S/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}\right)
= (52)(0.704183608830784014) - (50)(e^{-0.12(1/4)})(0.650426217823152353)
= 5.057386760.
\]

5. What is the price of a European call option on a non-dividend paying stock when the stock price is $30, the exercise price is $29, the risk-free interest rate is 5%, the volatility is 25% per annum, and the time to maturity is 4 months?

Solution: This can be evaluated in Maple with the commands, `with(finance);` and `evalf(blackscholes (30, 29, .05, 1/3, .25))` to obtain 2.52515.
In more detail,

\[ d_1 = \frac{\log\left(\frac{S}{K}\right) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]
\[ = \log\left(\frac{30}{29}\right) + (0.05 + 0.25^2/2)(1/3) \]
\[ = 0.4225156800 \]

\[ d_2 = \frac{\log\left(\frac{S}{K}\right) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}} \]
\[ = \log\left(\frac{30}{29}\right) + (0.05 - 0.25^2/2)(1/3) \]
\[ = 0.2781781127 \]

Then \( \Phi(0.4225156800) = 0.663675670914689153 \) and \( \Phi(0.2781781127) = 0.609562182219208660 \). Finally,

\[ S\Phi\left(\frac{\log(S/K) + (r + \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}\right) - Ke^{-r(T-t)}\Phi\left(\frac{\log(S/K) + (r - \sigma^2/2)(T - t)}{\sigma \sqrt{T - t}}\right) \]
\[ = (30)(0.663675670914689153) - (29)(e^{-0.05(1/3)})(0.609562182219208660) \]
\[ = 2.52514697. \]

6. Show by substitution that two exact solutions of the Black-Scholes equations are

(a) \( V(S,t) = AS, A \) some constant.

(b) \( V(S,t) = A \exp(rt) \)

Explain in financial terms what each of these solutions represents. That is, describe a simple “claim”, “derivative” or “option” for which this solution to the Black Scholes equation gives the value of the claim at any time.

Solution: The Black-Scholes equation is :

\[ V_t + \frac{1}{2}\sigma^2S^2V_{SS} + rSV_S - rV = 0 \]
(a) If \( V(S, t) = AS \), \( V_t = 0 \), \( V_S = A \) and \( V_{SS} = 0 \). Then substituting these into the Black-Scholes equation, we test to see if
\[
0 + \frac{1}{2} \sigma^2 S^2 \cdot 0 + rSA - rAS = 0
\]
and since it does, we see that \( V(t, S) = AS \) is a solution of the Black-Scholes equation.

(b) If \( V(S, t) = Ae^{rt} \), \( V_t = Are^{rt} \), \( V_S = 0 \) and \( V_{SS} = 0 \). Then substituting these into the Black-Scholes equation, we test to see if
\[
Are^{rt} + \frac{1}{2} \sigma^2 S^2 \cdot 0 + rS \cdot 0 - rAe^{rt} = 0
\]
and since it does, we see that \( V(t, S) = Ae^{rt} \) is a solution of the Black-Scholes equation.

(a) The portfolio containing solely \( A \) shares of the underlying security (and no bond or European option on the security) would have value \( AS \) and would be a solution of the Black-Scholes equation by construction.

(b) The portfolio containing solely \( A \) shares of the bond (and no security or option on the security) appreciating at market rate \( r \) would have value \( Ae^{rt} \) and would be a solution of the Black-Scholes equation by construction.

7. A pharmaceutical company has a stock that is currently $25. Early tomorrow morning the Food and Drug Administration will announce that it has either approved or disapproved for consumer use the company’s cure for the common cold. This announcement will either immediately increase the stock price by $10 or decrease the price by $10. Discuss the merits of using the Black-Scholes formula to value options on the stock.

Solution: No, the required assumptions in our mathematical model of the option value are:

(a) that a very large number of identical, rational traders always have complete information about all assets they are trading,
changes in prices may be random, but are continuous with some probability distribution,
that trading transactions take negligible time,
purchases and sales can be made in any amounts, that is, the stock and bond are divisible, we can buy them in any amounts including negative amounts (which are short positions),
the risky security issues no dividends.

All of the assumptions except the second are reasonably well met in this case. For instance, all traders know that an announcement affecting the stock’s value is imminent, meeting the requirement that the traders have complete information. However, the change in price should be assumed to be discontinuous, due to the sudden sharp change of 40% in value. The second assumption about continuous change is not well modeled. The conclusions of the Black-Scholes equation would not be justified in pricing an option on the company’s stock.

8. Use the put-call parity relationship to derive the relationship between

(a) The Delta of European call and the Delta of European put.
(b) The Gamma of European call and the Gamma of European put.

Solution: The put-call parity relation is

\[ S - V_C + V_P = K \exp(-r(T - t)). \]

(a) Differentiating both sides of this equation with respect to \( S \), we find, using the definition that \( \Delta_P = \frac{\partial V_P}{\partial S} \) and \( \Delta_C = \frac{\partial V_C}{\partial S} \),

\[ 1 - \Delta_C + \Delta_P = 0. \]

(b) Differentiating both sides of the previous equation with respect to \( S \), we find, using the definition that \( \Gamma_P = \frac{\partial^2 V_P}{\partial S^2} \) and \( \Gamma_C = \frac{\partial^2 V_C}{\partial S^2} \),

\[ -\Gamma_C + \Gamma_P = 0. \]