

Euclidean Geometry: A Review

We review some important concepts of Euclidean Geometry. We state most results without proof, but it is both instructive and challenging for you to think of why they are true.

Let us first talk of triangles. **Euclid's First Postulate** reassures us that two distinct points in the plane lie exactly on a line. What happens when we add a third point? Well, they may be **collinear** or they may not be collinear. If you take three noncollinear points in the plane, you can draw lines between them to produce a triangle.

How do we classify triangles?

We classify triangles according to the relative lengths of their sides and/or their angles. In a **scalene** triangle, all sides are of different lengths. If at least two of the three sides have equal length, we call it an **isosceles** triangle. Of course, the **equilateral** triangle is very special: in an equilateral, all sides have the same length.

Remark An equilateral triangle is also an isosceles triangle.

Question 1** A triangle in which all the angles have equal measure is an equiangular triangle. Is an equilateral triangle always equiangular? On the other hand, is an equiangular triangle always an equilateral?

Congruent Triangles

Congruence is an extremely useful notion to have! Try for a moment to think of some important results in geometry you have explained to your students using congruence of triangles. What is the philosophy behind the concept of congruence? Draw several triangles and then with a protractor and ruler, measure their angles and sides. What do you observe? In a given triangle, is the longest side opposite to the angle in the triangle with the largest measure?

We say that two line segments AB and CD are congruent if and only if they have the same length. Likewise, two angles are congruent if and only if they have the same measure.

Definition Two triangles $\triangle ABC$ and $\triangle DEF$ are congruent, written $\triangle ABC \cong \triangle DEF$ if there is a correspondence between them such that corresponding sides are congruent and corresponding angles are congruent.

Caveat! The word *correspondence* is of primary importance in this definition. We pair off the sides (and angles) of the two triangles according to whether they have the SAME MEASURE.

Now, how does one go about showing that two triangles are congruent? Here are the basic axioms and theorems.

SAS Axiom If two triangles $\triangle ABC$ and $\triangle DEF$ satisfy $\angle A \cong \angle D$, $AB \cong DE$ and $AC \cong DF$, then $\triangle ABC \cong \triangle DEF$.

ASA Theorem If two triangles $\triangle ABC$ and $\triangle DEF$ satisfy $\angle A \cong \angle D$, $AB \cong DE$ and $\angle B \cong \angle E$, then $\triangle ABC \cong \triangle DEF$.

SSS Theorem If two triangles $\triangle ABC$ and $\triangle DEF$ satisfy $AB \cong DE$ and $AC \cong DF$ and $BC \cong EF$, then $\triangle ABC \cong \triangle DEF$.

AAS Theorem If two triangles $\triangle ABC$ and $\triangle DEF$ satisfy $\angle A \cong \angle D$, $\angle B \cong \angle E$ and $BC \cong EF$, then $\triangle ABC \cong \triangle DEF$.

If I were to give you an isosceles triangle in which the two lateral sides are of equal measure, do you think you could assume that the two angles at the base of the triangle are also equal? Recall that two sides of a triangle are equal if and only if the angles opposite these sides are also equal. This is the well known **Isosceles Triangle Theorem**. How can you apply *congruent triangles* to explain why an isosceles triangle has two angles of equal measure?

Yet one more question! Does the *AAA* condition imply congruence? What can you say if *AAA* holds? Does the word *similar* come to mind? Let us wait a while before we get to that. I want to close this section on triangles with one very vital fact.

Angle Sum Theorem The sum of the measures of the angles in a triangle is equal to 180° .

Polygons There is nothing sacrosanct about two or three points; we want to throw in as many points as we like and join pairs of points. If we follow the rules of the game, we get polygons. These are 'closed curves' with each side lying on a straight line and no two distinct sides cross each other. The **quadrilaterals, parallelograms and trapezoids** are polygons having exactly four sides. What do you call a polygon with three sides? With five sides? With six sides?

A polygon in which all sides are congruent is said to be **regular**. Regular polygons are nice! Their innate symmetry has captured the imagination of mathematicians for ages. The Greeks as usual were no exceptions and neither are we!

Take a regular polygon. Now, draw lines joining pairs of vertices which are not already joined by a side. These are the **diagonals**. Extend any side of a regular polygon and notice that this line forms an **exterior angle** with the adjacent side of the polygon.

Fact The measure of an exterior angle of a regular polygon having n sides is $\frac{360^\circ}{n}$.

Two consecutive sides of a regular polygon form an angle, called an **interior angle** . Given a regular polygon with say, 5 sides, what is the measure of each of its interior angles? Are they all congruent?

Here are some useful facts about polygons which may not be regular.

1. A quadrilateral is a parallelogram (i.e. opposite sides are parallel) if and only if opposite sides are of equal length.
2. The diagonals of a parallelogram bisect each other.
3. All four angles of a rectangle are right angles.
4. Trapezoids are *almost* parallelograms. In a trapezoid, at least one pair of opposite sides are parallel.

Let us calculate the area enclosed by some of these polygons.

Area of rectangles and parallelograms For rectangles, the formula for area is simply, *base* \times *height*. Not surprisingly, the area of a parallelogram is given by the same expression, $A = b \times h$. As always, height is measured perpendicular to the base.

Area of a Triangle The area for a triangle looks surprising close to the area of a parallelogram! (Why?) The area of a triangle with base b and height h is given by $A = \frac{1}{2}(b \times h)$. You can now use this well known formula and Pythagoras Theorem to derive that the area of an equilateral triangle of side a is $\frac{\sqrt{3}a^2}{4}$.

Some more useful formulae

Area of a trapezoid with sides b_1, b_2 and height h is $A = \frac{1}{2}h(b_1 + b_2)$.

Area of a circle of radius r is πr^2 .

Question** Do congruent triangles (or polygons, for that matter) have the same area?

Its time for a quick foray into similarity Similarity is another important tool in geometry. Congruent figures can be rigidly moved to coincide with each other. However, similar figures have same shape but not have the same size. Therefore, while congruent figures are similar, similar figures are not necessarily congruent! The notion of similarity can be applied to any type of geometric object but we will confine ourselves to similar triangles. Once again, given two triangles $\triangle ABC$ and $\triangle DEF$, we need to find a correspondence

between their sides and angles. However, this time we demand that **corresponding angles of the triangles are congruent while the ratio of corresponding sides is the same for all such pairs.** We write $\triangle ABC \approx \triangle DEF$ and refer to the ratio, $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$ as the scale factor.

The SAS condition for similarity If two triangles $\triangle ABC$ and $\triangle DEF$ satisfy $\angle A \cong \angle D$ and $\frac{AB}{DE} = \frac{AC}{DF}$, then $\triangle ABC \approx \triangle DEF$.

Question Does similarity hold for triangles satisfying the AAA condition?

An Application of Similarity: Altitude of a Triangle An altitude of a triangle is a line segment that joins a vertex to the side opposite to it, and what is more, it is perpendicular to the opposite side. If two triangles are similar, with a scale factor of a , then their altitudes are also in the same ratio!

Comparing areas of similar figures Suppose two triangles are similar and the scale factor for their corresponding sides is a . The areas of the two triangles differ by a scale factor of a^2 .

Medians A median of a triangle is the line segment joining a vertex to the midpoint of the side opposite to it. It turns out that the three medians of a triangle intersect at a point, called the centroid of the triangle.

Important facts

1. A triangle is dissected by its medians into six smaller triangles of equal area.
2. The medians of a triangle divide one another in the ratio 2 : 1, in other words, the medians of a triangle *trisect* one another.