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Review prepared by: **Raegan Higgins** and **Silvia Saccon** (September 2006)
Revised by: **Silvia Saccon** (September 2007)

Algebra Review

1 Linear Equations in One Variable

Definition. A **linear equation in one variable** can be written in the form

$$Ax + B = C,$$

where A, B, C are real numbers with $A \neq 0$.

Solving a Linear Equation in One Variable

1. **Clear fractions.** Eliminate fractions by multiplying each side of the equation by the least common denominator.
2. **Simplify each side separately.** Use the distributive property to clear parentheses and combine like terms. **Distributive Property -** For any real numbers a, b, c,
$$a(b+c) = ab + ac \quad \text{and} \quad (b+c)a = ba + ca.$$
3. **Isolate the variable terms on one side.** Add and subtract to collect all variable terms on one side.
4. **Isolate the variable.** Divide both sides by the coefficient of the variable.
5. **Check.** Substitute your result from Step 4 in the given equation.

Example. Solve $4x - 2x - 5 = 4 + 6x + 3$.
Solution.

1. There are no fractions to clear.
2. $$\begin{aligned} 4x - 2x - 5 &= 4 + 6x + 3 \\ 2x - 5 &= 7 + 6x \end{aligned}$$
3. $$\begin{aligned} 2x - 5 + 5 &= 7 + 6x + 5 \\ 2x &= 12 + 6x \\ 2x - 6x &= 12 + 6x - 6x \\ -4x &= 12 \end{aligned}$$
4. $$\begin{aligned} \frac{-4x}{-4} &= \frac{12}{-4} \\ x &= -3 \end{aligned}$$

5.

$$\begin{aligned} 4(-3) - 2(-3) - 5 &= 4 + 6(-3) + 3 \\ -12 + 6 - 5 &= 4 - 18 + 3 \\ -11 &= -11 \end{aligned}$$

Hence $x = -3$ is the solution to the linear equation $4x - 2x - 5 = 4 + 6x + 3$.

Example. Solve $\frac{x+7}{6} + \frac{2x-8}{2} = -4$.
Solution.

1. The least common denominator of 6 and 2 is 6.

$$6\left(\frac{x+7}{6} + \frac{2x-8}{2}\right) = 6(-4)$$

2.

$$\begin{aligned} 6\left(\frac{x+7}{6} + \frac{2x-8}{2}\right) &= 6(-4) \\ (x+7) + 3(2x-8) &= -24 \\ x + 7 + 6x - 24 &= -24 \\ 7x - 17 &= -24 \end{aligned}$$

3.

$$\begin{aligned} 7x - 17 + 17 &= -24 + 17 \\ 7x &= -7 \end{aligned}$$

4.

$$\begin{aligned} \frac{7x}{7} &= \frac{-7}{7} \\ x &= -1 \end{aligned}$$

5.

$$\begin{aligned} \frac{-1+7}{6} + \frac{2(-1)-8}{2} &\stackrel{?}{=} -4 \\ \frac{6}{6} + \frac{-10}{2} &\stackrel{?}{=} -4 \\ 1 - 5 &\stackrel{?}{=} -4 \\ -4 &\stackrel{\checkmark}{=} -4 \end{aligned}$$

Hence $x = -1$ is the solution to the linear equation $\frac{x+7}{6} + \frac{2x-8}{2} = -4$.

2 Systems of Linear Equations in Two Variables

A system of equations is composed of two or more equations considered simultaneously.

2.1 Solving a Linear System by Elimination

1. **Write both equations in standard form** $Ax + By = C$.
2. **Make the coefficients of one pair of variable terms opposites.** Multiply one or both equations by appropriate numbers so that the sum of the coefficients of either the x- or y-terms is 0.
3. **Add** the new equations to eliminated a variable. The sum should be an equation with just one variable.
4. **Solve** the equation from Step 3 for the remaining variable.
5. **Find the other value.** Substitute the result of Step 4 into either of the original equations and solve for the other variable.
6. **Check** the solution in both of the original equations. Then write the solution set.

Example. Solve the system

$$\begin{cases} 5x - 2y = 4 & (1) \\ 2x + 3y = 13 & (2) \end{cases}.$$

Solution.

1. Both equations are in standard form.
2. Suppose we want to eliminate the variable x. One way to do this is to multiply equation (1) by 2 and equation (2) by -5.
$$\begin{aligned} 10x - 4y &= 8 \\ -10x - 15y &= -65 \end{aligned}$$
3. Now add.
$$\begin{array}{rcr} 10x - 4y &=& 8 \\ -10x - 15y &=& -65 \\ \hline -19y &=& -57 \end{array}$$
4. Solve for y.
$$\begin{aligned} \frac{-19y}{-19} &= \frac{-57}{-19} \\ y &= 3 \end{aligned}$$
5. To find x, substitute 3 in for y in either equation (1) or (2). Substituting in equation (2) gives
$$\begin{aligned} 2x + 3(3) &= 13 \\ 2x &= 4 \\ x &= 2. \end{aligned}$$
6. The solution is (2,3). To check, substitute 2 for x and 3 for y in both equations (1) and (2).
$$\begin{aligned} 5x - 2y &\stackrel{?}{=} 4 \\ 5(2) - 2(3) &\stackrel{?}{=} 13 \\ 10 - 6 &\stackrel{?}{=} 4 \\ 4 &\stackrel{\checkmark}{=} 4 \end{aligned} \qquad \begin{aligned} 2x + 3y &\stackrel{?}{=} 13 \\ 2(2) + 3(3) &\stackrel{?}{=} 13 \\ 4 + 9 &\stackrel{?}{=} 13 \\ 13 &\stackrel{\checkmark}{=} 13 \end{aligned}$$

The solution set is $\{(2,3)\}$.

2.2 Solving a Linear System by Substitution

1. **Solve one of the equations for either variable.** (If one of the variable terms has coefficient 1 or -1, choose it since the substitution method is usually easier this way.)

2. **Substitute** for that variable in the other equation. The result should be an equation in just one variable.

3. **Solve** the equation from Step 2 for the remaining variable.

4. **Find the other value.** Substitute the result of Step 3 into the equation from Step 1 to find the value of the other variable.

5. **Check** the solution in both of the original equations. Then write the solution set.

Example. Solve the system

$$\begin{cases} 3x + 2y = 13 & (1) \\ 4x - y = -1. & (2) \end{cases}.$$

Solution.

1. Solve one of the equations for either x or y. Since the coefficient of y in equation (2) is -1, it is easiest to solve for y in equation (2).

$$\begin{aligned} 4x - y &= -1 \\ -y &= -1 - 4x \\ y &= 1 + 4x \end{aligned}$$

2. Substitute $1 + 4x$ for y in equation (1).

$$\begin{aligned} 3x + 2y &= 13 \\ 3x + 2(1 + 4x) &= 13 \end{aligned}$$

3. Solve for x.

$$\begin{aligned} 3x + 2 + 8x &= 13 \\ 11x &= 11 \\ x &= 1 \end{aligned}$$

4. Solve for y.

$$\begin{aligned} y &= 1 + 4x \\ y &= 1 + 4(1) \\ y &= 5 \end{aligned}$$

5. Check the solution (1,5) in both equations (1) and (2).

$$\begin{aligned} 3x + 2y &\stackrel{?}{=} 13 \\ 3(1) + 2(5) &\stackrel{?}{=} 13 \\ 3 + 10 &\stackrel{?}{=} 13 \\ 13 &\stackrel{\checkmark}{=} 13 \end{aligned} \qquad \begin{aligned} 4x - y &\stackrel{?}{=} -1 \\ 4(1) - 5 &\stackrel{?}{=} -1 \\ 4 - 5 &\stackrel{?}{=} -1 \\ -1 &\stackrel{\checkmark}{=} -1 \end{aligned}$$

The solution set is $\{(1,5)\}$.

3 The Quadratic Equation

Definition. A **quadratic equation** is an equation that can be written in the form

$$ax^2 + bx + c = 0,$$

where a, b, c are real numbers with $a \neq 0$.

Quadratic Formula
The solutions of $ax^2 + bx + c = 0$ ($a \neq 0$) are given by

$$x_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad x_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a} \, .$$

Example. Solve $6x^2 - 5x - 4 = 0$.
Solution. Here $a = 6$, $b = -5$, and $c = -4$. So the solutions to $6x^2 - 5x - 4 = 0$ are:

$$x_1 = \frac{5 + \sqrt{(-5)^2 - 4(6)(-4)}}{2(6)} = \frac{5 + \sqrt{25 + 96}}{12} = \frac{5 + 11}{12} = \frac{16}{12} = \frac{4}{3}$$

and

$$x_2 = \frac{5 - \sqrt{(-5)^2 - 4(6)(-4)}}{2(6)} = \frac{5 - \sqrt{25 + 96}}{12} = \frac{5 - 11}{12} = -\frac{6}{12} = -\frac{1}{2} \, .$$

Example. Solve $p^2 + \dfrac{p}{3} = \dfrac{1}{6}$.
Solution. First we must this equation in standard form $ap^2 + bp + c = 0$ ($a \neq 0$). So we clear the denominators by multiplying both sides of the equation by the least common denominator of 3 and 6, which is 6.

$$6\left(p^2 + \frac{p}{3} = \frac{1}{6}\right) \quad \Longrightarrow \quad 6p^2 + 2p = 1 \, .$$

Write the equation in standard form:

$$6p^2 + 2p - 1 = 0 \, .$$

Hence $a = 6$, $b = 2$, and $c = -1$. So the solutions to the equation $6p^2 + 2p - 1 = 0$ are given by:

$$\begin{aligned} p &= \boxed{} \\ &= \vdots \\ &= \frac{-2 \pm \sqrt{28}}{12} = \frac{-1 \pm \sqrt{7}}{6} \, . \end{aligned}$$

Some useful formulas for factoring quadratic polynomials

1. $x^2 - 2xb + b^2 = (x - b)^2$
2. $x^2 + 2xb + b^2 = (x + b)^2$
3. $x^2 - b^2 = (x + b)(x - b)$

4 Radicals and Factorization

4.1 Rationalizing Fractions

Example. Simplify $\frac{1}{\sqrt{2}}$.
Solution. The goal is to get rid of the square root in the denominator. We do this by multiplying both the numerator and the denominator by $\sqrt{2}$.

$$\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

In general, for $a > 0$ we have:

$$\frac{1}{\sqrt{a}} = \frac{\sqrt{a}}{a}.$$

Example. Simplify $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}}$.
Solution. We want to get rid of the square roots in the denominator. We shall do this by using the rule $(a^2 - b^2) = (a+b)(a-b)$.

$$\frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} \cdot \frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}} = \frac{\sqrt{2}(\sqrt{2}-\sqrt{3})}{2-3} = -\sqrt{2}(\sqrt{2}-\sqrt{3}) = -2+\sqrt{6}$$

Some useful rules for simplifying radicals

1. $\sqrt{a}\sqrt{b} = \sqrt{ab}$
2. $\sqrt{a^2} = |a|$
3. $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$

4.2 Simplifying radicals

Example. Simplify $\sqrt{220}$.
Solution. In order to simplify, first factor 220 into a product of prime powers.

$$220 = 2^2 \cdot 5 \cdot 11$$

Since $\sqrt{2^2} = 2$, we have

$$\sqrt{220} = 2\sqrt{5 \cdot 11} = 2\sqrt{55}.$$

Since 5 and 11 are both prime numbers, we can't simplify anymore.

Example. Simplify $\sqrt{75600}$.
Solution. We have $75600 = 7 \cdot 5^2 \cdot 3^3 \cdot 2^4$, so

$$\sqrt{75600} = 2^2 \cdot 3 \cdot 5\sqrt{7 \cdot 3} = 60\sqrt{21}.$$

5 Factorials

Example. We have 5 different mathematic books to give out. Cheryl, Steve, Nathan, Christina and Silvia each want a book. If you give each person exactly one math book, in how many ways can you hand out the books?
Solution. Label the books A, B, C, D and E. We have 5 choices of whom to give book A. After we give away book A, there are only 4 people left. So we have 4 choices of whom to give book B. That leaves 3 choices for whom to give book C. After we give away book C, there are 2 people left without books. This leaves 2 choices of whom to give book D. Then we are left with book E and one person which whom to give it to.

Now we need to use the Fundamental Counting Principle:

Given a combined action, or event, in which the first action can be performed in n_1 ways, the second action can be performed in n_2 ways, and so on, the total number of ways in which the combined action can be performed is the product

$$n_1 \cdot n_2 \cdot \cdots \cdot n_k \ .$$

So we have a total of $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways of handing out the books.
We will often use products such as $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$ and so it is convenient to adopt a notation for them. For the product

$$5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$$

we write 5!, read "5 factorial".

In general, given n distinct objects to arrange, there are

$$n! = n \cdot (n-1) \cdot (n-2) \cdots 2 \cdot 1$$

ways to arrange the objects.
(*Terminology.* A **permutation** of a set of n objects is an ordered arrangement of all n objects.)

Remarks on Factorial Notation.

- For the number 0, $0! = 1$.
- For any natural number n, $n! = n \cdot (n-1)!$.

Example. How many ways are there to arrange the letter of the word "question"?
Solution. There are 8 different letters in the word question. Thus using the formula above we see that there are $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 40320$ ways to arrange the letters of the word "question".

6 Complex Fractions

A **complex fraction** is an expression having a fraction in the numerator, denominator or both.

Example. Simplify $\dfrac{\frac{r^2-4}{4}}{1+\frac{2}{r}}$.

Solution (Method 1).

1. Simplify the numerator and the denominator separately.

$$\frac{\frac{r^2-4}{4}}{1+\frac{2}{r}} = \frac{\frac{(r-2)(r+2)}{4}}{\frac{r+2}{r}}$$

2. Divide by multiplying the numerator by the reciprocal of the denominator.

$$\begin{aligned}\frac{\frac{r^2-4}{4}}{1+\frac{2}{r}} &= \frac{\frac{(r-2)(r+2)}{4}}{\frac{r+2}{r}} \\ &= \frac{(r-2)(r+2)}{4} \div \frac{r+2}{r} \\ &= \frac{(r-2)(r+2)}{4} \cdot \frac{r}{r+2}\end{aligned}$$

3. Simplify the resulting fraction (if possible).

$$\begin{aligned}\frac{\frac{r^2-4}{4}}{1+\frac{2}{r}} &= \frac{\frac{(r-2)(r+2)}{4}}{\frac{r+2}{r}} \\ &= \frac{(r-2)(r+2)}{4} \div \frac{r+2}{r} \\ &= \frac{(r-2)(r+2)}{4} \cdot \frac{r}{r+2} \\ &= \frac{r(r-2)}{4}\end{aligned}$$

Example. Simplify $\dfrac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}$.

Solution (Method 2).

1. Multiply the numerator and the denominator of the complex fraction by the least common denominator of the fractions in the numerator and the fractions in the denominator of the complex fraction.

   The LCD of all the fractions in the numerator and the denominator of the complex fraction is LCD=x^2y^2.

$$\begin{aligned}\frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}} &= \frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}\cdot 1 = \frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}}\cdot\frac{x^2y^2}{x^2y^2}\\ &= \frac{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)\cdot x^2y^2}{\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)\cdot x^2y^2}\\ &= \frac{xy^2+x^2y}{y^2-x^2}\end{aligned}$$

2. Simplify the resulting fraction (if possible).

$$\begin{aligned}\frac{\dfrac{1}{x}+\dfrac{1}{y}}{\dfrac{1}{x^2}-\dfrac{1}{y^2}} &= \frac{\left(\dfrac{1}{x}+\dfrac{1}{y}\right)\cdot x^2y^2}{\left(\dfrac{1}{x^2}-\dfrac{1}{y^2}\right)\cdot x^2y^2}\\ &= \frac{xy^2+x^2y}{y^2-x^2}\\ &= \frac{xy(y+x)}{(y+x)(y-x)}\\ &= \frac{xy}{y-x}\end{aligned}$$

7 Continued Fractions

Example. Simplify

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}.$$

Solution. We start from the bottom and work up. So the first step is to replace $1+\dfrac{1}{1}$ with 2.

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}$$

Continuing to work our way up we get,

$$1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}}=1+\cfrac{1}{1+\cfrac{1}{\frac{3}{2}}}=1+\cfrac{1}{1+\cfrac{2}{3}}=1+\cfrac{1}{\frac{5}{3}}=1+\frac{3}{5}=\frac{8}{5}.$$

Example. Simplify

$$1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9}}}}.$$

Solution.

$$\begin{aligned}
1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{7+\cfrac{8}{9}}}} &= 1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{6}{\frac{71}{9}}}}\\
&= 1+\cfrac{2}{3+\cfrac{4}{5+\cfrac{54}{71}}}\\
&= 1+\cfrac{2}{3+\cfrac{4}{\frac{409}{71}}}\\
&= 1+\cfrac{2}{3+\cfrac{284}{409}}\\
&= 1+\cfrac{2}{\frac{1511}{409}}\\
&= 1+\frac{818}{1511}\\
&= \frac{2329}{1511}
\end{aligned}$$

8 Functions

Definition. A **function** f from a set A to a set B is a rule that assigns to each element x in the set A exactly one element y in the set B. We write $y = f(x)$, where $f(x)$ is the *value of the function f at x* (function notation).
We say x is the **independent variable** and y is the **dependent variable**.
The set A is the **domain** of the function f, i.e. the domain is the set of all values of the independent variable for which the function is defined (set of inputs).
The set B is called the **codomain** of the function f (set of all possible outputs).
The set of all values $f(x)$ (in B) is called the **range** of the function f, i.e. the range is the set of all values assumed by the dependent variable. The range of f is always a subset of the codomain B of f.

Function Notation
When an equation is used to represent a function, it is convenient to name the function so that it can be referenced easily. For example, the equation $y = 1 - x^2$ describes the variable y as a function of the variable x. We use the following **function notation**.

Input	*Output*	*Equation*
x	$f(x)$	$f(x) = 1 - x^2$

The symbol $f(x)$ corresponds to the y-value for a given x. So we can write $y = f(x)$.

Characteristics of a Function f from Set A to Set B

(1) Each element in A must be matched with any element in B.

(2) An element in A (the domain) cannot be matched with two different elements of B.

(3) Some elements in B may not be matched with any element in A. Two or more elements in A may be matched with the same element in B.

Example. Which of the equations represent(s) y as a function of x?

(a) $x^2 + y = 1$ (b) $-x + y^2 = 1$

Solution. To determine whether y is a function of x, try to solve for y in terms of x.

(a) Solve for y:
$$x^2 + y = 1 \quad \Longrightarrow \quad y = 1 - x^2 \ .$$
To each value of x there corresponds exactly one value of y. So y is a function of x.

(b) Solve for y:
$$-x + y^2 = 1 \quad \Longrightarrow \quad y^2 = 1 + x \quad \Longrightarrow \quad y = \pm\sqrt{1+x} \ .$$
The $\pm$ indicates that to a given value of x there correspond two values of y. So y is not a function of x.

Example. Which sets of ordered pairs represent functions from A to B, where $A = \{0, 1, 2, 3\}$ and $B = \{-2, -1, 0, 1, 2\}$? Why or why not?

(a) $\mathcal{C}_1 = \{(0, 1), (1, -2), (2, 0), (3, 2)\}$

(b) $\mathcal{C}_2 = \{(0, -1), (2, 2), (1, -2), (3, 0), (1, 1)\}$

(c) $\mathcal{C}_3 = \{(0, 2), (3, 0), (1, 1)\}$

(d) $\mathcal{C}_4 = \{(0, 0), (1, 0), (2, 0), (3, 0)\}$

Solution.

(a) The set $\mathcal{C}_1$ does represent a function from A to B because each element x in A is matched to exactly one element y in B.

(b) The set $\mathcal{C}_2$ does not represent a function from A to B because the element 1 in A corresponds to two different elements, -2 and 1, in B.

(c) The set $\mathcal{C}_3$ does not represent a function from A to B because not every element in A is matched with an element in B.

(d) The set $\mathcal{C}_4$ does represent a function from A to B because every element in A is matched with an element in B. Regardless of the value of $x \in A$, the value of $y \in B$ is always 0. Such functions are called *constant functions.*

Example (Evaluating a Function). Find $f(-1)$, $f(0)$ and $f(2)$, where

$$f(x) = \begin{cases} -x^2 + 1 & \text{if } x < 0 \\ 2x - 1 & \text{if } x \geq 0 \end{cases} .$$

Solution. Since $x = -1$ is less that 0, use $f(x) = -x^2 + 1$ to obtain

$$f(-1) = -(-1)^2 + 1 = 0 .$$

For $x = 0$, use $f(x) = 2x - 1$ to obtain

$$f(0) = 2 \cdot 0 - 1 = -1 .$$

And for $x = 2$ use $f(x) = 2x - 1$ to obtain

$$f(2) = 2 \cdot 2 - 1 = 3 .$$

9 Sequences

Definition. A **sequence** is a function whose domain is the set of integers starting with some integer n_0 (often 0 or 1).

Example. The function $a(n) = \frac{1}{n}$, for $n = 1, 2, \ldots$ defines the sequence

$$1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$$

Here 1 is called the first term, $\frac{1}{2}$ is the second term and so on.

We call $a(n) = \frac{1}{n}$ the **general term**, since it gives a (general) formula for computing all the terms of the sequence.

- We usually use subscript notation instead of function notation and write a_n instead of $a(n)$.

Example. Consider the sequence $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, \cdots$.
This sequence is called the *Fibonacci sequence*. We denote the first term of the sequence by F_1, the second term by F_2 and more generally the n-th term by F_n (or $F(n)$). For example $F_1 = 1$, $F_3 = 2$ and $F_7 = 13$.
The Fibonacci sequence is given by

$$\begin{cases} F_1 = 1, \\ F_2 = 1, \\ F_n = F_{n-1} + F_{n-2} \quad \text{for } n > 2. \end{cases}$$

Example. In the sequence $2, 3, 5, 7, 11, 13, 17, \cdots$, where we denote the n-th term by p_n, what is p_3? p_5?
Solution. Since the third and fifth term of the sequence are 5 and 11 respectively, we have $p_3 = 5$ and $p_5 = 11$.

Example. Suppose $a_1 = 1$ and $a_n = 2a_{n-1}$ for $n > 1$. Find a_3 and a_{100}.
Solution. Write out the first few terms of the sequence. We have:

$$\begin{aligned} a_1 &= 1, \\ a_2 &= 2(1) = 2, \\ a_3 &= 2(2) = 4, \\ a_4 &= 2(4) = 8, \\ a_5 &= 2(8) = 16, \\ &\vdots \end{aligned}$$

So we see that $a_3 = 4$.
Now to find a_{100} we could write out all 100 terms, but that would be tedious. If we look at the terms of the sequence that we have already written out, we notice that

$$\begin{aligned} a_1 &= 1 = 2^0, \\ a_2 &= 2 = 2^1, \\ a_3 &= 4 = 2^2, \\ a_4 &= 8 = 2^3, \\ &\vdots \end{aligned}$$

Based on these observations, we see that the general term a_n is given by

$$a_n = 2^{n-1}, \quad n \geq 1 .$$

Thus $a_{100} = 2^{99}$.

10 Set Theory

Set Theory deals with the properties of collections, or **sets**, of objects - the **elements** of the set - conceived as a whole.

Notation	*Example*	*Reads*
$\{,\ldots,\}$	$\{x_1, x_2, x_3, \ldots x_n\}$	"Set with elements $x_1, x_2, \ldots, x_n$"
$\in$	$x \in A$	"x belongs to A" or "x is an element of the set A
$\notin$	$x \notin A$	"x does not belong to A" or "x is not an element of the set A
$\subseteq$	$B \subseteq A$	"B is a subset of A"
$\cup$	$A \cup B$	"The union of A and B"
$\cap$	$A \cap B$	"The intersection of A and B"
$\setminus$	$A \setminus B$	"The difference of A and B" or A minus B
^{C}	A^C	"The complement of A (in U)"

- One set may be an element of another set, e.g. $C = \{1, 2, \{3, 4\}, 5\}$ has an element that is a set, namely $\{3, 4\}$, and we write $\{3, 4\} \in C$.

  Example. In the sets

  $$A = \{1, 2, 3\} \qquad B = \{1, \{2, 3\}\} \qquad C = \{\{1, 2\}, 3\}\ ,$$

  1 is an element of A and B but not of C, i.e. $1 \in A$, $1 \in B$, $1 \notin C$.

- The **empty set** $\emptyset$ contains no elements.

- A **finite set** is empty or contains a finite number of elements.

  A **infinite set** contains an infinite number of elements. The number sets $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{R}$ are all infinite.

- If each element of a set A is contained in a set B, then A is a **subset** of B.

- Every set is a subset of the **universal set** U, which contains all elements capable of being accepted to the problem.

  Also, every set is a subset of itself and the empty set is a subset of itself.

  Example. Consider the set $A = \{3\}$. Then A has one element, 3, and two subsets, $\emptyset$ and the set itself:

  $3 \in \{3\}$ $\quad \{3\} \subseteq \{3\}$ $\quad 3 \not\subseteq \{3\}$ $\quad \emptyset \subseteq \{3\}$ $\quad \emptyset \notin \{3\}$

- A **power set** is the set of all subsets of a given set, containing the empty set and the original set.

  Example. If $A = \{a, b, c\}$, then $\mathcal{P}(A) = \{\emptyset, \{a\}, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{c, b\}, \{a, b, c\}\}$.

  How many elements does $\mathcal{P}(A)$ contain? If the number of elements of A is n, then $\mathcal{P}(A)$ has 2^n elements.

10.1 Operations with Sets

- The **union** $A \cup B$ of sets A and B is the set of all elements that appear at least once in the original sets A, B.

  Example. If $A = \{x \in \mathbb{R} | -2 < x \leq 4\}$ and $B = \{x \in \mathbb{R} | x > 0\}$, then
  $A \cup B = \{x \in \mathbb{R} | x > -2\}$.

- The **intersection** $A \cap B$ of sets A and B is the set of all elements that are common to the original sets A and B.

  Example. If $A = \{x \in \mathbb{R} | -2 < x \leq 4\}$ and $B = \{x \in \mathbb{R} | x > 0\}$, then
  $A \cap B = \{x \in \mathbb{R} | 0 < x \leq 4\}$.

- The **difference** $A \setminus B$ of sets A and B is the set of all elements that belong to A but not to B.

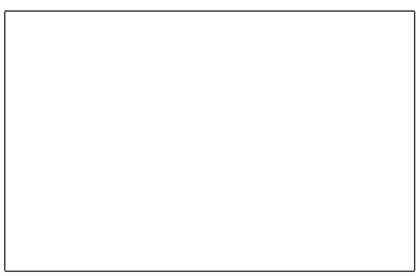
  Example. If $A = \{1, 2, 4, 5\}$ and $B = \{2, 5\}$, then $A \setminus B = \{1, 4\}$ and $B \setminus A = \emptyset$.

- The **complement** A^C of set A is the difference between the universal set and the subset A, i.e. $A^C = U \setminus A$.

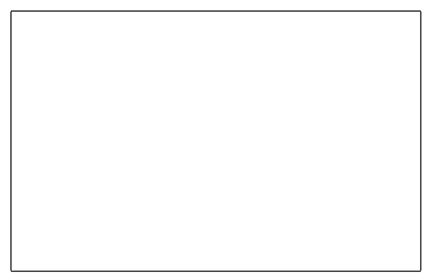
  Example. If the universal set is $U = \mathbb{R}$ and $A = \{x \in \mathbb{R} | -2 < x \leq 4\} \subseteq U$, then
  $A^C = \{x \in \mathbb{R} | x \leq -2 \text{ or } x > 4\}$.

10.2 Venn Diagrams

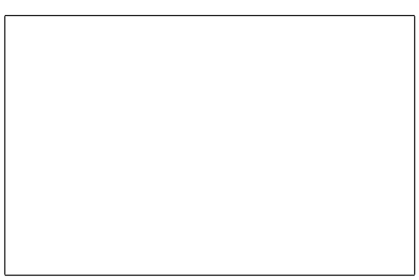
A **Venn Diagram** is a rectangle - the universal set U - that includes circles depicting the subsets.



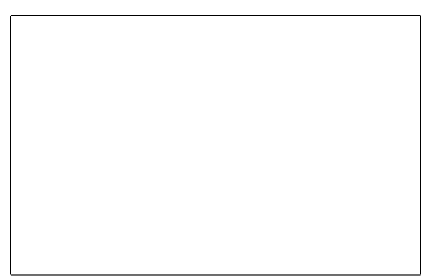
Union. The shaded area represents $A \cup B$.



Intersection. The shaded area represents $A \cap B$.



Difference. The shaded area represents $A \setminus B$.



Complement. The shaded area represents A^C.

Example. Of 160 individuals, 100 had been exposed to chemical **A** (individuals A), 50 to chemical **B** (individuals B) and 30 to chemicals **A** and **B**. Use Venn Diagrams to describe the number of individuals exposed to

(1) chemicals **A** and chemical **B**

(2) chemical **A** but not chemical **B**

(3) chemical **A** or chemical **B**

(4) neither chemical **A** nor chemical **B**

Solution. Of individuals A, 100-30=70 had been exposed only to chemical **A**. Of individuals B, $50 - 30 = 20$ had been exposed only to chemical **B**.

(1) The number of individuals exposed to both chemical **A** and chemical **B** can be expressed as $|A \cap B| = 30$.

(2) The number of individuals exposed to chemical **A** but not chemical **B** can be expressed as $|A \setminus B| = |A \cap B^C| = 70$.

(3) The number of individuals exposed to chemical **A** or chemical **B** can be expressed as $|A \cup B| = 120$.

(4) The number of individuals exposed to neither chemical **A** nor chemical **B** can be expressed as $|A^C \cap B^C| = |(A \cup B)^C| = 40$.

10.3 Algebra of Sets

There are similarities but also significant differences between conventional algebra and the algebra of sets. "Prove" the following identities using Venn Diagrams!

- Commutative Laws
  1. $A \cup B = B \cup A$
  2. $A \cap B = B \cap A$
- Idempotent Laws: $A \cup A = A = A \cap A$
- Associative Laws
  1. $A \cup (B \cup C) = (A \cup B) \cup C$
  2. $A \cap (B \cap C) = (A \cap B) \cap C$
- Distributive Laws
  1. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
  2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- Idempotent Laws: $A \cup \emptyset = A$ and $A \cap \emptyset = \emptyset$
- de Morgan's Laws
  1. $(A \cup B)^C = A^C \cap B^C$
  2. $(A \cap B)^C = A^C \cup B^C$

Problem. 800 individuals were examined for antigens called α_1, α_2 and α_3.
Number of individual positive for

α_1	500
α_2	350
α_3	400
α_1 and α_2	250
α_2 and α_3	150
α_1 and α_3	200
α_1, α_2 and α_3	50

Find the following.

(1) The number of individuals negative for all three antigens, α_1, α_2 and α_3.

(2) The number of individuals positive for antigen α_2 but negative for both α_1 and α_3.

Hint. Use Venn Diagrams!
Solution. (1) 100 (2) 0