Steven R. Dunbar
Department of Mathematics
203 Avery Hall
University of Nebraska-Lincoln
Lincoln, NE 68588-0130
http://www.math.unl.edu
Voice: 402-472-3731
Fax: 402-472-8466

Question of the Day

Key Concepts

1. Bernstein Motion
2.
3.

Vocabulary

1. centered Gaussian random variable
2. covariance matrix
3. Gaussian stochastic process

4. convergence in distribution

Mathematical Ideas

This section is adapted from: "Bernstein Polynomials and Brownian Motion", by Emmanuel Kowalski, American Mathematical Monthly, December 2006, pages 865-886.

Bernstein Motions

Definition 1 We say that a real-valued random variable X is a *centered Gaussian random variable* with variance σ^2 if it has a probability density function

$$\frac{1}{\sigma\sqrt{2\pi}} \exp(-t^2/2\sigma^2)$$

or equivalently for any chasuble set $B \subset \mathbb{R}$

$$\mathbb{P}[X \in B] = \int_B \frac{1}{\sigma\sqrt{2\pi}} \exp(-t^2/2\sigma^2)\, dt.$$

Lemma 1 If X is a centered Gaussian random variable with variance parameter σ^2, then

1. $$\mathbb{E}[X] = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} t \exp(-t^2/2\sigma^2)\, dt = 0.$$

2. $$\operatorname{Var} X = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} t^2 \exp(-t^2/2\sigma^2)\, dt = \sigma^2.$$

3.

$$\mathbb{E}\left[X^4\right] = \frac{1}{\sigma\sqrt{2\pi}} \int_{\mathbb{R}} t^4 \exp(-t^2/2\sigma^2)\, dt = 3 \operatorname{Var} X^2 = 3\sigma^2.$$

Definition 2 The *characteristic function* of a random variable X is

Definition 3 We say the random vector $(X_1, \ldots, X_m)$ is a *centered Gaussian random variable* if

$$\mathbb{P}\left[(X_1, \ldots X_m) \in B\right] = \frac{1}{\sqrt{|\det(A)|}(2\pi)^{m/2}} \in_B exp(-\sum_{i,j} b_{ij} t_i t_j)\, dt_1 \ldots dt_m$$

where $\mathbb{E}\left[X_i X_j\right] = a_{ij}$ are the entries of the *covariance matrix* A, and (b_{ij}) is the inverse of A.

Note that this is the m-dimensional extension of the well-known distribution of a (one-dimensional) Gaussian (or normally-distributed) random variable

$$\mathbb{P}\left[X \in B\right] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_B exp(-t^2/\sigma^2)\, dt.$$

Definition 4 A mapping $x \mapsto X_t$, where X_t is a random variable for each t (usually $t \in [0, 1]$ or $t \in [0, \infty)$) is a *Gaussian stochastic process parametrized by t* if the vector $(X_{t_1}, \ldots, X_{t_m})$ is a centered Gaussian random variable for each finite subset $(t_1, \ldots, t_m)$. The covariance function of the stochastic process is

$$g(s, t) = \mathbb{E}\left[X_s X_t\right]$$

We define a *Bernstein Motion* in the following way. For each positive integer n let $(Y_{nj})_{1 \leq j \leq n}$ be a vector of independent, centered, Gaussian random variables with variance $1/n$. (Also, later we will want to refer to the probability space Ω which is rich enough to support all the random variables Y_{nj}, but for now we will suppress the underlying probability space.)

Define;

$$X_{n0} = 0,$$
$$X_{nj} = Y_{n1} + \cdots + Y_{nj}, 1 \leq j \leq n$$
$$\mathcal{B}_n(x) = \sum_{j=0}^{n} \binom{n}{j} X_{nj} x^j (1-x)^{n-j}$$

Consider the properties of the Bernstein motion analogous to Brownian motion:

1. At $x = 0$ all the Bernstein polynomials except $b_{n,0}(x) = (1-x)^n$ evaluate to 0. Then $\mathcal{B}_n(0) = X_{n0}(1-0)^n = 0$.

2. For each value $x \in [0,1]$, $\mathcal{B}_n(x)$is a finite linear combination of Gaussian random variables, so the distribution of $\mathcal{B}_n(x)$ is Gaussian. Therefore, $x \mapsto \mathcal{B}_n(x)$ is a Gaussian process. Then in particular, If $0 \leq x_1 < x_2 \ldots < x_n$, then the vector $(B(x_1), \ldots, B(x_n))$, is a centered Gaussian random vector

3. The first step is to compute the covariance function $\mathbb{E}\left[\mathcal{B}_n(x)\mathcal{B}_n(y)\right]$.

   Lemma 2 The covariance function of $\mathcal{B}_n$ satisfies

   $$\mathbb{E}\left[\mathcal{B}_n(x)\mathcal{B}_n(y)\right] = Q_n(x,y)$$

   where Q_n is the nth Bernstein polynomial of the continuous function $\min(x,y)$ on $[0,1] \times [0,1]$.

   Proof: By the definition of the Bernstein motions and the linearity of the expectation

$$\mathbb{E}\left[\mathcal{B}_n(x)\mathcal{B}_n(y)\right] = \sum_j \sum_k \binom{n}{j}\binom{n}{k} x^j(1-x)^{n-j} y^k (1-y)^{n-k} \mathbb{E}\left[X_{nj}X_{nk}\right]$$

$$= \sum_j \sum_k \binom{n}{j}\binom{n}{k} x^j(1-x)^{n-j} y^k (1-y)^{n-k} \sum_{1\le j'\le j}\ \sum_{1\le k'\le k} \mathbb{E}\,[$$

$$= \sum_j \sum_k \binom{n}{j}\binom{n}{k} x^j(1-x)^{n-j} y^k (1-y)^{n-k} \sum_{1\le j'\le j}\ \sum_{1\le k'\le k} \frac{j',}{n}$$

In the inner double sum, the only terms that contribute are when $j' = k'$, and they can only contribute for $j' \le \min(j,k)$. Therefore

$$\mathbb{E}\left[X_{nj}X_{nk}\right] = \frac{\min(j,k)}{n} = \min(j/n, k/n).$$

Then

$$\mathbb{E}\left[\mathcal{B}_n(x)\mathcal{B}_n(y)\right] = \sum_j \sum_k \binom{n}{j}\binom{n}{k} \min j/n, k/n x^j(1-x)^{n-j} y^k(1-y)^{n-k}$$

and the lemma is established.

Corollary 1

- For each $x \in [0,1]$ the sequence of random variables $(\lfloor_n(x))$ converges in distribution to a centered Gaussian random variable with variance x.
- For each (x,y) in $[0,1]^2$, the sequence of vectors $(\mathcal{B}_n(x)\mathcal{B}_n(y))$ converges in distribution to a vector $(B(x), B(y))$of centered Gaussian random variables with $\mathbb{E}\left[B(x)B(y)\right] = \min(x,y)$.

From this corollary we see that if we suggestively denote $\lim n \to \infty = B(x)$, then the resulting stochastic process is a Gaussian process satisfying all the required properties of Brownian motion except the continuity of sample paths. The problem is that there is no a priori relation between X_{nj} and X_{mk} for $n \neq m$. Yet according to the heuristic construction we should have X_{nj} represents $B(j/n) = B(k/m)$ for instance if $k = jl$ and $m = ln$.

Convergence to Wiener Measure

Recall that $C([0,1], \mathbb{R})$ is a complete metric space with metric given by the norm $\|f\| = \max_{[0,1]} |f(x)|$. Let $\mathcal{B}$ be the associated Borel σ-algebra of subsets of $C([0,1], \mathbb{R})$. Recall that a Borel σ-algebra is the smallest σ-algebra containing the open sets in the topology of the space.

Lemma 3 The σ-algebra $\mathcal{B}$ is generated by the cylinder sets

$$C_{x,A} = \{f \in C([0,1], \mathcal{R}) : f(x) \in A\}$$

for $x \in [0,1]$ and A a Borel subset of $\mathcal{R}$.

Remark 1 One could call this a "gate" set as well as a cylinder set, since $C_{x,A}$ requires the functions f pass through the gate A at the "time" x.

Remark 2 Note that if A is an open set in $\mathbb{R}$, then $C_{x,A}$ is an open set in the (metric-space) topology of $C([0,1], \mathbb{R})$. One can see this in two ways:

1. Let $f \in C_{x,A}$ so $f(x) \in A$, and there is an $\epsilon > 0$, so that if $|y - f(x)| \leq \epsilon$, then $y \in A$. Now, consider g such that

$\|g - f\| = \max_{z\in[0,1]} |g(z) - f(z)| \le \epsilon$. Then all the more so, $|g(x) - f(x)| \le \epsilon$, and so $g(x) \in A$.

2. The "evaluation at x" map $f \mapsto f(x)$ is continuous for the reason in the previous item. Hence the pre-image in $C([0,1],\mathbb{R})$ of A under this evaluation map is an open set.

Proof: I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer Verlag, New York, 1988, Chapter 2, Section 4, Exercise 4.2.

Now consider $B_n^{-1}(C_{x,A})$. This is $\{\omega : \mathcal{B}_n(\omega, x) \in A\}$ which is measurable by definition for fixed x.

Example 1 As an almost trivial example, take $n = 2$, $x = 1/3$, $A = (1,2)$. Then $\mathcal{B}_2(x) = X_{20}x^2 + 2X_{21}x(1-x) + X_{22}(1-x)^2 = 2Y_{n1}x(1-x) + (Y_{21} + Y_{22})(1-x)^2$ recalling that $X_{20} = 0$ and (Y_{21}, Y_{22}) is a pair of independent centered Gaussian random variables with variance $1/n = 1/2$. We will look at $\mathcal{B}_2^{-1}(C_{x,A}) = \{\omega : 1 < 2Y_{n1} \cdot (2/9) + (Y_{21} + Y_{22})(4/9) < 2\} = \{\omega : 1 < (8/9) \cdot Y_{21} + (4/9) \cdot Y_{22} < 2\}$. This is clearly an event on the probability space for the random variables Y_{nj}. Note that

We will use the Bernstein motions $\mathcal{B}_n$ to define measures W_n on $C([0,1],\mathbb{R})$. For a Borel set X of functions in $C([0,1],\mathbb{R})$, let Y be the set of events in Ω for which the corresponding nth-degree Bernstein *motion* is in X. Then let $W_n(X) = \mathbb{P}[Y]$. We are going to show that the sequence of probability measures W_n converges in distribution to a W which is Wiener measure.

Example 2 As an almost trivial example, take $n = 2$, $x = 1/3$, $A = (1,2)$. Let X be the open set $C_{1/3,(1,2)}$, hence X is a Borel set. Then we showed above that $Y = \{\omega : 1 < (8/9) \cdot Y_{21} + (4/9) \cdot Y_{22} < 2\}$. Just for fun, we know that $(8/9) \cdot Y_{21} + (4/9) \cdot Y_{22}$

is a centered Gaussian random variable with variance $(8/9)^2 \cdot (1/2) + (4/9)^2 \cdot (1/2) = (80/162)$. Then

$$\mathbb{P}\left[1 < (8/9) \cdot Y_{21} + (4/9) \cdot Y_{22} < 2\right] = 0.0751511988$$

On the other hand, the Wiener measure of $C_{1/3,(0,1)}$ is $\mathbb{P}\left[1 < \mathcal{N}(0,1/3) < 2\right] = 0.0413662555$

Theorem 1 With $\mathcal{B}_n$ defined as above, the sequence (W_n) of probability measures on $C([0,1],\mathbb{R})$ converges in distribution to a probability measure W such that on the probability space $C([0,1],\mathbb{R},\mathcal{B},W)$ the stochastic process defined by the random variables $B(x)$ where $B(x)(f) = f(x)$ for f in $C([0,1],\mathbb{R})$ is a Brownian motion restricted to $[0,1]$

In order to prove the theorem we first need the foll wing application of Skorohod's criterion for convergence of probability measures in distribution:

Theorem 2 Let $(\mathcal{B}_n)$ be a sequence of continuous Gaussian processes on $[0,1]$ with the property that for each finite set $t_1 < t_2 < \cdots < t_m$ in $[0,1]$ the Gaussian random vectors $(\mathcal{B}_1, \ldots, \mathcal{B}_N(t_m))$ converge in distribution. Suppose that $\mathcal{B}_n(0) = 0$ almost surely, and that

$$\mathbb{E}\left[(\mathcal{B}_n(x) - \mathcal{B}_n(y))^4\right] \leq C|x-y|^2$$

holds on $[0,1]^2$ for some constant C. Then the associated sequence of probability measures $W_n = \mathbb{P}\left[\mathcal{B}_n\right]$ on $C([0,1],\mathbb{R})$ converges in distribution to the random variable $f \mapsto f(x)$ on the probability space $C([0,1],\mathbb{R})$ equipped with the measure W.

Proof: I. Karatzas and S. Shreve, *Brownian Motion and Stochastic Calculus*, Springer Verlag, New York, 1988, Chapter 2, Section 4, Exercise 4.11, Theorem 4.15.

Problems to Work for Understanding

1.
2.
3.
4.

Reading Suggestion:

1.
2.
3.

Outside Readings and Links:

1.
2.
3.
4.

I check all the information on each page for correctness and typographical errors. Nevertheless, some errors may occur and I would be grateful if you would alert me to such errors. I make every reasonable effort to present current and accurate information for public use, however I do not guarantee the accuracy or timeliness of information on this website. Your use of the information from this website is strictly voluntary and at your risk.

I have checked the links to external sites for usefulness. Links to external websites are provided as a convenience. I do not endorse, control, monitor, or guarantee the information contained in any external website. I don't guarantee that the links are active at all times. Use the links here with the same caution as you would all information on the Internet. This website reflects the thoughts, interests and opinions of its author. They do not explicitly represent official positions or policies of my employer.

Information on this website is subject to change without notice.

Last modified: Processed from LaTeX source on February 4, 2009