MULTIPLICITIES AND ENUMERATION OF SEMIDUALIZING MODULES

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ABSTRACT. A finitely generated module C over a commutative noetherian ring R is semidualizing if $\mathrm{Hom}_R(C,C) \cong R$ and $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$. For certain local Cohen-Macaulay rings $(R,\mathfrak{m})$, we verify the equality of Hilbert-Samuel multiplicities $e_R(J;C) = e_R(J;R)$ for all semidualizing R-modules C and all $\mathfrak{m}$-primary ideals J. The classes of rings we investigate include those that are determined by ideals defining fat point schemes in projective space or by monomial ideals. We use these ideas to show that if R is local (or graded) and Cohen-Macaulay with codimension 2, then R has at most two (graded) semidualizing modules, up to isomorphism, namely R and a dualizing module.

1. Introduction

In this section, let $(R,\mathfrak{m},k)$ be a Cohen-Macaulay local ring with a dualizing module D. A finitely generated R-module C is *semidualizing* if $\mathrm{Hom}_R(C,C) \cong R$ and $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$. Thus, the module D is precisely a semidualizing module of finite injective dimension. Let $\mathfrak{S}_0(R)$ denote the set of isomorphism classes of semidualizing R-modules. (See Section 2 for definitions and background information.) For example, the R-modules R and D are semidualizing. The ring R is Gorenstein if and only if $D \cong R$, equivalently, if and only if $\mathfrak{S}_0(R) = \{[R]\}$.

One reason to study semidualizing modules can be found in their application to the study of the following:

Question 1.1 (Huneke)**.** If R is not Gorenstein, must the sequence of Betti numbers $\{\beta^R_i(D)\}$ be unbounded?

In [13, (3.6)] the following is proved: If there is a semidualizing R-module C such that $R \ncong C \ncong D$, then the sequence of Betti numbers $\{\beta^R_i(D)\}$ is bounded below by a linear polynomial in i and is hence unbounded. Because of this result, one need only answer Huneke's question in the case when R does not admit a semidualizing module C such that $R \ncong C \ncong D$, that is, when $\mathfrak{S}_0(R) = \{[R],[D]\}$. Accordingly, we are interested in solving the following:

Problem 1.2. Characterize the rings R such that $\mathfrak{S}_0(R) = \{[R],[D]\}$.

Toward this goal, the following result is contained in Theorem 5.2. The graded version is Corollary 5.4.

Theorem 1.3. *If R has codimension 2, then* $\mathfrak{S}_0(R) = \{[R],[D]\}$.

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Recall that $\operatorname{codim}(R) = \operatorname{edim}(R) - \dim(R)$ where $\operatorname{edim}(R)$ is the minimal number of generators of $\mathfrak{m}$. Rings of codimension 0 are regular, and codimension 1 rings are hypersurfaces. In particular, if $\operatorname{codim}(R) \leqslant 1$, then R is Gorenstein so $\mathfrak{S}_0(R) = \{[R]\}$. Hence, the first potentially interesting case is codimension 2.

It is worth noting that Huneke's question has been answered in the affirmative for rings of codimension 2 by Leuschke and Jorgensen [12]; see also the work of Christensen, Striuli, and Veliche [5]. Also, note that there exist rings of codimension greater than 2 such that $\mathfrak{S}_0(R) = \{[R], [D]\}$; see, e.g. Lemma 3.1.

Our proof of Theorem 1.3 hinges on the computation of $\operatorname{len}_{R_P}(C_P)$ where $[C] \in \mathfrak{S}_0(R)$ and $P \in \operatorname{Ass}(R)$, which we do in Lemma 5.1. In a sense, this is equivalent to computing the Hilbert-Samuel multiplicity $e_R(C)$, thus justifying our interest in the following question, motivated by the well-known equality $e_R(J; D) = e_R(J; R)$:

Question 1.4. Let C be a semidualzing R-module. For each $\mathfrak{m}$-primary ideal J, must we have an equality of Hilbert-Samuel multiplicities $e_R(J; C) = e_R(J; R)$?

When R is generically Gorenstein (e.g., reduced) an affirmative answer to this question is contained in [14, (2.8(a))]. In Theorems 3.2 and 3.4 we address a few more special cases with the following:

Theorem 1.5. *Assume that R satisfies one of the following conditions:*

(1) $P^2 R_P = 0$ *for each* $P \in \operatorname{Ass}(R)$*;*
(2) $\widehat{R} \cong k[\![X_0, X_1, \cdots, X_n]\!]/Ik[\![X_0, X_1, \cdots, X_n]\!]$ *where* $I \subseteq k[X_0, X_1, \cdots, X_n]$ *is the ideal determining a fat point scheme in* $\mathbb{P}^n_k$*; or*
(3) $\widehat{R} \cong k[\![X_1, \ldots, X_n]\!]/I$ *where I is generated by monomials in the* X_i.

For every $\mathfrak{m}$-primary ideal $J \subset R$ and every semidualizing R-module C, we have $e_R(J; C) = e_R(J; R)$.

Note that we make no assumption about the codimension of R in Theorem 1.5.

This paper is organized as follows. Section 2 consists of background material, and Section 3 contains the proof of Theorem 1.5. Section 4 is concerned with the connection between lengths and Betti numbers of semidualizing modules. The paper concludes with the proof of Theorem 1.3 in Section 5.

2. Background

For the rest of this paper, let R and S be commutative noetherian rings of finite Krull dimension.

Definition 2.1. Let C be an R-module. The natural *homothety map*

$$\chi^R_C \colon R \to \operatorname{Hom}_R(C, C)$$

is the R-module homomorphism given by $\chi^R_C(r)(c) = rc$. The module C is *semidualzing* if it satisfies the following:

(1) C is finitely generated;
(2) the homothety map $\chi^R_C \colon R \to \operatorname{Hom}_R(C, C)$ is an isomorphism; and
(3) $\operatorname{Ext}^i_R(C, C) = 0$ for all $i > 0$.

The module C is *dualizing* if it is semidualizing and has finite injective dimension.[1]

[1]The assumption $\dim(R) < \infty$ guarantees that a finitely generated R-module C has finite injective dimension over R if and only if $C_\mathfrak{m}$ has finite injective dimension over $R_\mathfrak{m}$. For instance,

Example 2.2. It is straightforward to show that the free R-module R^1 is semidualizing. It is dualizing if and only if R is Gorenstein.

The following facts will be used in the sequel.

Fact 2.3. If C is a cyclic semidualizing module, then the equalities $0 = \operatorname{Ker}(\chi_C^R) = \operatorname{Ann}_R(C)$ imply that $C \cong R$.

Fact 2.4. Let C be a semidualizing R-module. Then a sequence $x_1, \ldots, x_n \in R$ is C-regular if and only if it is R-regular. (See, e.g., [13, (1.4)] for a brief explanation of the local case. The general case has the same proof.)

Fact 2.5. Assume that R is Cohen-Macaulay and that D is a dualizing R-module. Let C be a semidualizing R-module. From [6, (3.1), (3.4)] and [9, (V.2.1)], we have the following:

(a) $\operatorname{Ext}^i_R(C, D) = 0$ for all $i \geqslant 1$;
(b) the dual $\operatorname{Hom}_R(C, D)$ is a semidualizing R-module;
(c) the natural biduality map $\delta_C^D \colon C \to \operatorname{Hom}_R(\operatorname{Hom}_R(C, D), D)$ given by the formula $\delta_C^D(c)(\psi) = \psi(c)$ is an isomorphism;
(d) $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, D)) = 0$ for all $i \geqslant 1$; and
(e) the natural evaluation map $C \otimes_R \operatorname{Hom}_R(C, D) \to D$ given by $c \otimes \psi \mapsto \psi(c)$ is an isomorphism.

From (c), we conclude that:

(f) if $\operatorname{Hom}_R(C, D) \cong R$, then $C \cong D$.

Assume that R is local. Because of (d) and (e), the minimal free resolution of D is obtained by tensoring the minimal free resolutions of C and $\operatorname{Hom}_R(C, D)$. In particular, this implies that:

(g) $\beta_i^R(D) = \sum_{j=0}^{i} \beta_j^R(C)\beta_{i-j}^R(\operatorname{Hom}_R(C, D))$ for each $i \geqslant 0$.

Fact 2.6. Let $\varphi \colon R \to S$ be a homomorphism of commutative noetherian rings. Assume that S has finite flat dimension as an R-module. For example, this is satisfied when S is flat as an R-mdoule, or when φ is surjective with $\operatorname{Ker}(\varphi)$ generated by an R-regular sequence. If C is a semidualizing R-module, then $S \otimes_R C$ is a semidualizing S-module; the converse holds when φ is faithfully flat; see [6, (4.5)]. Thus, the rule of assignment $[C] \mapsto [S \otimes_R C]$ describes a well-defined function $\mathfrak{S}_0(\varphi) \colon \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$. If the map φ is local, that is if $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are local and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, then the induced map $\mathfrak{S}_0(\varphi)$ is injective; see [6, (4.9)].

Assume that φ is local and satisfies one of the following conditions:

(1) φ is flat with Gorenstein closed fibre $S/\mathfrak{m}S$ (e.g., φ is the natural map from R to its completion $\widehat{R}$); or
(2) φ is surjective with $\operatorname{Ker}(\varphi)$ generated by an R-regular sequence.

Then a semidualizing R-module C is dualizing for R if and only if $S \otimes_R C$ is dualizing for S by [2, (3.1.15),(3.3.14)]. When R is complete and φ satisfies condition (2), the induced map $\mathfrak{S}_0(\varphi) \colon \mathfrak{S}_0(R) \to \mathfrak{S}_0(S)$ is bijective; see [7, (4.2)] or [8, (2)].

Fact 2.7. Assume that $(R, \mathfrak{m}, k)$ is local and C is a semidualizing R-module. If C has finite projective dimension, then $C \cong R$; see, e.g., [13, (1.14)]. If R is

this removes the need to worry about any distinction between the terms "dualizing" and "locally dualizing", and similarly for "Gorenstein" and "locally Gorenstein". This causes no loss of generality in this paper as we are primarily concerned with local and graded situations.

Gorenstein, then $C \cong R$ by [4, (8.6)]. If $\mathfrak{m}^2 = 0$, then either $C \cong R$ or C is dualizing for R. (Indeed, if $C \ncong R$, then the first syzygy C' of C is a non-zero k-vector space such that $\operatorname{Ext}^1_R(C', C) = 0$, so C is injective.)

The following notions are standard.

Remark/Definition 2.8. Let $(R, \mathfrak{m})$ be a local ring and let I be an $\mathfrak{m}$-primary ideal of R. Let C be a finitely generated R-module of dimension d. There is a polynomial $H_{I,C}(j) \in \mathbb{Q}[j]$ such that $H_{I,C}(j) = \operatorname{len}_R(I^jC/I^{j+1}C)$ for $j \gg 0$. Moreover, the degree of $H_{I,C}(j)$ is $d-1$, and the leading coefficient is of the form $e_R(I;C)/(d-1)$ for some positive integer $e_R(I;C)$. The integer $e_R(I;C)$ is the *Hilbert-Samuel* multiplicity of C with respect to I.

The next lemma is a version of a result of Herzog [11, (2.3)]. It is proved similarly and is almost certainly well-known.

Lemma 2.9. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such that $\mathfrak{m}S = \mathfrak{n}$. Let I be an $\mathfrak{m}$-primary ideal of R, and let C be a finitely generated R-module. Set $\widetilde{C} = S \otimes_R C$ and $\widetilde{I} = IS$. For each j there is an equality $\operatorname{len}_S(\widetilde{I}^j\widetilde{C}/\widetilde{I}^{j+1}\widetilde{C}) = \operatorname{len}_R(I^jC/I^{j+1}C)$. In particular, we have $e_S(\widetilde{I};\widetilde{C}) = e_R(I;C)$.*

We end this section with a discussion of the graded situation.

Fact 2.10. Assume that $R = \coprod_{i\geqslant 0} R_i$ is graded where R_0 is local with maximal ideal $\mathfrak{m}_0$. Set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geqslant 1} R_i$, and let C be a finitely generated graded R-module. Then C is a (semi)dualizing R-module if and only if $C_\mathfrak{m}$ is a (semi)dualizing $R_\mathfrak{m}$-module by [6, (2.15)]. If R is Cohen-Macaulay, then it has a graded dualizing module that is unique up to homogeneous isomorphism; see [2, (3.6.9)–(3.6.12)].

The next three lemmas allow us to pass between local and graded situations.

Lemma 2.11. *Assume that $R = \coprod_{i\geqslant 0} R_i$ is graded where R_0 is local with maximal ideal $\mathfrak{m}_0$. Set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geqslant 1} R_i$, and let C be a graded semidualizing R-module. The following conditions are equivalent:*

(i) $C \cong R$;
(ii) $C_\mathfrak{m} \cong R_\mathfrak{m}$;
(iii) $\operatorname{pd}_R(C) < \infty$; *and*
(iv) $\operatorname{pd}_{R_\mathfrak{m}}(C_\mathfrak{m}) < \infty$.

Proof. The implications (i) $\Longrightarrow$ (ii) $\Longrightarrow$ (iv) are routine. The equivalence (ii) $\Longleftrightarrow$ (iv) if contained in Fact 2.7, and the equivalence (iii) $\Longleftrightarrow$ (iv) follows from the equality $\operatorname{pd}_{R_\mathfrak{m}}(C_\mathfrak{m}) = \operatorname{pd}_R(C)$; see [2, (1.5.15(e))].

(ii) $\Longrightarrow$ (i) Assume that $C_\mathfrak{m} \cong R_\mathfrak{m}$. In particular, the $R_\mathfrak{m}$-module $C_\mathfrak{m}$ is cyclic. From [2, (1.5.15(a))], we conclude that C is a cyclic R-module, so $C \cong R$ by Fact 2.3. $\square$

The ideas for the following lemma are contained in [2, Sec. 3.6], though it is not stated there explicitly.

Lemma 2.12. *Assume that $R = \coprod_{i\geqslant 0} R_i$ is graded where R_0 is local with maximal ideal $\mathfrak{m}_0$. Set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geqslant 1} R_i$, and let C be a finitely generated graded R-module. Then $\operatorname{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) < \infty$ if and only if $\operatorname{id}_R(C) < \infty$.*

Proof. One implication follows from the inequality $\mathrm{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) \leqslant \mathrm{id}_R(C)$.

For the other implication, assume that $d = \mathrm{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) < \infty$. Let ${}^*\mathrm{id}_R(C)$ denote the length of a minimal *injective resolution of C; see [2, Sec. 3.6]. Because of the inequality $\mathrm{id}_R(C) \leqslant {}^*\mathrm{id}_R(C) + 1$ from [2, (3.6.5(a))], it suffices to show that ${}^*\mathrm{id}_R(C) < \infty$. As a consequence of [2, (3.6.4)], we need only show that the Bass number $\mu_{d+1}(\mathfrak{p}, C) = 0$ for each graded prime ideal $\mathfrak{p} \subset R$. For each such $\mathfrak{p}$, our assumptions on R imply that $\mathfrak{p} \subseteq \mathfrak{m}$. This yields the first step in the next sequence

$$\mu_{d+1}^R(\mathfrak{p}, C) = \mu_{d+1}^{R_\mathfrak{m}}(\mathfrak{p}_\mathfrak{m}, C_\mathfrak{m}) = 0$$

while the second step is from the definition of d. □

Lemma 2.13. *Assume that $R = \coprod_{i\geqslant 0} R_i$ is graded where R_0 is local with maximal ideal $\mathfrak{m}_0$. Set $\mathfrak{m} = \mathfrak{m}_0 + \coprod_{i\geqslant 1} R_i$, and let C be a finitely generated graded R-module. Then C is dualizing for R if and only if $C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$.*

Proof. Fact 2.6 implies that C is semidualizing for R if and only if $C_\mathfrak{m}$ is semidualizing for $R_\mathfrak{m}$, and Lemma 2.12 says that $\mathrm{id}_{R_\mathfrak{m}}(C_\mathfrak{m}) < \infty$ if and only if $\mathrm{id}_R(C) < \infty$. This yields the desired result. □

Definition 2.14. Let k be a field. Fix distinct points $Q_1, \ldots, Q_r \in \mathbb{P}_k^n$ and integers $m_1, \ldots, m_r \geqslant 1$. Set $S_0 = k[X_0, X_1, \ldots, X_n]$ with irrelevant maximal ideal $\mathfrak{n}_0 = (X_0, X_1, \ldots, X_n)S_0$. For each index j, let $I(Q_j) \subset S_0$ be the (reduced) vanishing ideal of Q_j. The subscheme of $\mathbb{P}_k^n$ defined by the ideal $I = \cap_{j=1}^r I(Q_j)^{m_j} \subseteq S_0$ is the *fat point scheme* determined by the points $Q_1, \ldots, Q_r$ with multiplicities $m_1, \ldots, m_r$.

Remark 2.15. Continue with the notation of Definition 2.14.

Set $S = k[\![X_0, X_1, \cdots, X_n]\!]$ with maximal ideal $\mathfrak{n} = (X_0, X_1, \cdots, X_n)S$. The local rings $(S_0)_{\mathfrak{n}_0}/I_{\mathfrak{n}_0}$ and $R = S/IS$ are Cohen-Macaulay of dimension 1.

Note that the quotient $S_0/I(Q_j)$ is isomorphic (as a graded k-algebra) to a polynomial ring $k[Y]$. In particular, the completion of the local ring $(S_0)_{\mathfrak{n}_0}/I(Q_j)(S_0)_{\mathfrak{n}_0}$ (isomorphic to $S/I(Q_j)S$) is isomorphic to the formal power series ring $k[\![Y]\!]$. In particular, the ideal $I(Q_j)S$ is prime. It follows that the associated primes of $R = S/I$ are of the form $P_j = I(Q_j)S/I$. Localizing at one of these primes yields

$$R_{P_j} \cong S_{I(Q_j)}/IS_{I(Q_j)} \cong S_{I(Q_j)}/I(Q_j)^{m_j} S_{I(Q_j)} \cong S_{I(Q_j)}/(I(Q_j)S_{I(Q_j)})^{m_j}.$$

In other words, we have $R_{P_j} \cong S_j/\mathfrak{n}_j^{m_j}$ for some regular local ring $(S_j, \mathfrak{n}_j)$.

3. Multiplicities of Semidualizing Modules

In this section, we consider Question 1.4 for certain classes of rings.

Lemma 3.1. *Let $(S, \mathfrak{n})$ be a regular local ring containing a field. Let e be a positive integer, and set $R = S/\mathfrak{n}^e$. Let C be a semidualizing R-module. Then either $C \cong R$ or C is dualizing for R. In particular, we have $\mathrm{len}_R(C) = \mathrm{len}_R(R)$.*

Proof. Fact 2.7 deals with the case $e = 1$, so assume that $e \geqslant 2$. The ring R is artinian and local. Hence, it is complete and has a dualizing module D. There are isomorphisms

$$R \cong \widehat{R} \cong \widehat{S}/\mathfrak{n}^e\widehat{S} \cong k[\![X_1, \ldots, X_n]\!]/(X_1, \ldots, X_n)^e$$

where k is a field and $n = \mathrm{edim}(S)$. We now conclude from [14, (4.11)] that $C \cong R$ or $C \cong D$. The conclusion $\mathrm{len}_R(C) = \mathrm{len}_R(R)$ follows from the well-known equality $\mathrm{len}_R(D) = \mathrm{len}_R(R)$. □

The next result contains cases (1) and (2) of Theorem 1.5 from the introduction.

Theorem 3.2. *Let $(R, \mathfrak{m})$ be a local Cohen-Macaulay ring, and let C be a semidualizing R-module. Assume that R satisfies one of the following conditions:*

1. $P^2 R_P = 0$ *for each* $P \in \mathrm{Ass}(R)$*;*
2. $\widehat{R} \cong k[\![X_0, X_1, \cdots, X_n]\!]/Ik[\![X_0, X_1, \cdots, X_n]\!]$ *where* $I \subseteq k[X_0, X_1, \cdots, X_n]$ *is the ideal determining a fat point scheme in* $\mathbb{P}^n_k$*.*

Then for every $\mathfrak{m}$-primary ideal $J \subset R$, we have $e_R(J; C) = e_R(J; R)$.

Proof. (1) Assume that $P^2 R_P = 0$ for each $P \in \mathrm{Ass}(R)$. Fact 2.7 implies that $\mathrm{len}_{R_P}(C_P) = \mathrm{len}_{R_P}(R_P)$ for each $P \in \mathrm{Ass}(R)$, hence the second equality in the following sequence wherein each sum is taken over all $P \in \mathrm{Ass}(R)$:

$$e(J; C) = \sum_P \mathrm{len}_{R_P}(C_P) e(J; R/P) = \sum_P \mathrm{len}_{R_P}(R_P) e(J; R/P) = e(J; R).$$

The remaining equalities follow from the additivity formula [2, (4.7.t)].

(2) Using Fact 2.6 and Lemma 2.9 we may pass to the completion to assume that $R \cong \widehat{R}$. For each $P \in \mathrm{Ass}(R)$, Remark 2.15 implies that $R_P \cong S/\mathfrak{n}^m$ for some regular local ring $(S, \mathfrak{n})$. Lemma 3.1 implies that $\mathrm{len}_{R_P}(C_P) = \mathrm{len}_{R_P}(R_P)$ for each $P \in \mathrm{Ass}(R)$, hence the desired conclusion follows as in case (1). □

The next result contains part of case (3) of Theorem 1.5 from the introduction. The general case is in Theorem reflem204.

Lemma 3.3. *Let $(A, \mathfrak{r})$ be a complete reduced local ring, and set $S = A[\![x_1, \ldots, x_n]\!]$, the formal power series ring, with maximal ideal $\mathfrak{n} = (\mathfrak{r}, x_1, \ldots, x_n)S$. Let $I \subset S$ be an ideal generated by monomials in the x_i, and set $R = S/I$ with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. Assume that R is Cohen-Macaulay, and let C be a semidualizing R-module. Then for each $P \in \mathrm{Spec}(R)$ and for each PR_P-primary ideal $J \subset R_P$, we have $e_{R_P}(J; C_P) = e_{R_P}(J; R_P)$.*

Proof. Here is an outline of the proof. We show that the theory of polarization for monomial ideals yields a complete reduced Cohen-Macaulay local ring R^* and a surjection $\tau \colon R^* \to R$ such that $\mathrm{Ker}(\tau)$ is generated by an R^*-regular sequence $\mathbf{y}$. Facts 2.4 and 2.6 yield a semidualizing R^*-module such that the sequence $\mathbf{y}$ is C^*-regular and $C^*/(\mathbf{y})C^* \cong C$. Because R^* is complete and reduced, the desired conclusion follows from [14, (2.8.b)].

Set $S_0 = A[x_1, \ldots, x_n] \subset S$. The ideal $I_0 = I \cap S_0$ is generated by monomials in the x_i, in fact, by the same list of monomial generators used to generate I.

The theory of polarization for monomial ideals yields the following:

1. a polynomial ring $S_0^* = A[x_{1,1}, \ldots, x_{1,t_1}, x_{2,1}, \ldots, x_{2,t_2}, \ldots, x_{n,1}, \ldots, x_{n,t_n}]$ with irrelevant maximal ideal $\mathfrak{n}_0^* = (\mathfrak{r}, \{x_{i,j}\})S_0^*$,
2. an ideal $I_0^* \subseteq S_0^*$ generated by *square-free* monomials in the $x_{i,j}$,
3. a sequence $\mathbf{y} = y_1, \ldots, y_r \in \mathfrak{n}_0^*$ that is both S_0^*-regular and (S_0^*/I_0^*)-regular and such that $S_0^*/(\mathbf{y})S_0^* \cong S_0$ and $S_0^*/(I_0^* + (\mathbf{y})S_0^*) \cong S_0/I_0$.

Localizing at $\mathfrak{n}_0^*$ and passing to the completion yields the following:

(1′) a power series ring $S^* = A[\![x_{1,1}, \dots, x_{1,t_1}, x_{2,1}, \dots, x_{2,t_2}, \dots, x_{n,1}, \dots, x_{n,t_n}]\!]$ over k with maximal ideal denoted $\mathfrak{n}^* = (\mathfrak{r}, \{x_{i,j}\})S^*$,
(2′) an ideal $I^* = I_0^* S^* \subseteq S^*$ generated by square-free monomials in the $x_{i,j}$,
(3′) a sequence $\mathbf{y} = y_1, \dots, y_r \in \mathfrak{n}^*$ that is both S^*-regular and (S^*/I^*)-regular and such that $S^*/(\mathbf{y})S^* \cong S$ and $S^*/(I^* + (\mathbf{y})S^*) \cong S/I = R$.

Setting $R^* = S^*/I^*$, we have the following:

(1″) Since I^* is generated by square-free monomials, the ring R^* is reduced.
(2″) The sequence $\mathbf{y}$ is R^*-regular such that $R^*/(\mathbf{y})R^* \cong R$. In particular, since R is Cohen-Macaulay, so is R^*.
(3″) Since S^* is complete, so is R^*. Thus, Fact 2.6 provides a semidualizing R^*-module C^* such that $C \cong C^* \otimes_{R^*} R$.
(4″) Since the sequence $\mathbf{y}$ is R^*-regular, it is also C^*-regular by Fact 2.4.

Let $\tau\colon R^* \to R$ be the canonical surjection, and set $P^* = \tau^{-1}(P)$. We then have the following:

(1‴) Since R^* is reduced, so is the localization $R^*_{P^*}$.
(2‴) Since R^* is Cohen-Macaulay, so is the localization $R^*_{P^*}$. In particular, the ring $R^*_{P^*}$ is equidimensional. Also, we have $R^*_{P^*}/(\mathbf{y})R^*_{P^*} \cong R_P$.
(3‴) Since R^* is complete, it is excellent, and it follows that the localization $R^*_{P^*}$ is also excellent. In particular, for every $\mathfrak{p} \in \operatorname{Min}(R^*_{P^*})$ the ring $(R^*_{P^*})_{\mathfrak{p}}/\mathfrak{p}(R^*_{P^*})_{\mathfrak{p}} \otimes_{R^*_{P^*}} \widehat{R^*_{P^*}}$ is Gorenstein.
(4‴) The $R^*_{P^*}$-module $C^*_{P^*}$ is semidualizing and satisfies $C^*_{P^*}/(\mathbf{y})C^*_{P^*} \cong C_P$.

Using the conditions (1‴)—(4‴), the conclusion $e_{R_P}(J; C_P) = e_{R_P}(J; R_P)$ now follows from [14, (2.8.b)]. □

The next result contains case (3) of Theorem 1.5 from the introduction. In preparation, recall that a prime ideal P in a local ring R is *analytically unramified* if the completion $\widehat{R/P}$ is reduced. For example, if R is excellent, then every prime ideal of R is analytically unramified.

Theorem 3.4. *Let $(A, \mathfrak{r})$ be a complete reduced local ring, and $S = A[\![x_1, \dots, x_n]\!]$ the formal power series ring, with maximal ideal $\mathfrak{n} = (\mathfrak{r}, x_1, \dots, x_n)S$. Let $I \subset S$ be an ideal generated by monomials in the x_i. Let R be a local Cohen-Macaulay ring such that $\widehat{R} \cong S/I$, and let C be a semidualizing R-module.*

(a) *Let $P \in \operatorname{Spec}(R)$ be analytically unramified. Then for every PR_P-primary ideal $J \subset R_P$, we have $e_{R_P}(J; C_P) = e_{R_P}(J; R_P)$.*
(b) *For every $\mathfrak{m}$-primary ideal $J \subset R$, we have $e_R(J; C) = e_R(J; R)$.*

Proof. (a) Since the natural map $R \to \widehat{R}$ is flat and local, there is a prime $\widetilde{P} \in \operatorname{Spec}(\widehat{R})$ such that $P = \widetilde{P} \cap R$ and that the induced map $R_P \to \widehat{R}_{\widetilde{P}}$ is flat and local.

The R_P-module C_P is semidualizing. Furthermore, by flat base-change, the $\widehat{R}_{\widetilde{P}}$-module $\widehat{R}_{\widetilde{P}} \otimes_{R_P} C_P$ is semidualizing. The fact that P is analytically unramified implies that the maximal ideal of R_P extends to the maximal ideal of $\widehat{R}_{\widetilde{P}}$. Hence, Lemma 2.9 yields the first and third equalities in the following sequence:

$$e_{R_P}(J; C_P) = e_{\widehat{R}_{\widetilde{P}}}(J\widehat{R}_{\widetilde{P}}; \widehat{R}_{\widetilde{P}} \otimes_{R_P} C_P) = e_{\widehat{R}_{\widetilde{P}}}(J\widehat{R}_{\widetilde{P}}; \widehat{R}_{\widetilde{P}}) = e_{R_P}(J; R_P).$$

The second equality is from Lemma 3.3.

(b) Since $R/\mathfrak{m}$ is a field, it is complete. Hence, the prime ideal $\mathfrak{m}$ is analytically unramified, so the desired conclusion follows from part (a). □

Corollary 3.5. *Let $(S,\mathfrak{n})$ be a regular local ring containing a field, and let $\mathbf{x} = x_1,\ldots,x_n$ be a regular system of parameters for S. Let $I \subset S$ be an ideal generated by monomials in the x_i, and set $R = S/I$ with maximal ideal $\mathfrak{m} = \mathfrak{n}/I$. Assume that R is Cohen-Macaulay, and let C be a semidualizing R-module. For every $P \in \operatorname{Ass}(R)$, we have $\operatorname{len}_{R_P}(C_P) = \operatorname{len}_{R_P}(R_P)$.*

Proof. Since R is Cohen-Macaulay, we have $P \in \operatorname{Min}(R)$. This explains the first and third equalities in the next sequence:

$$\operatorname{len}_{R_P}(C_P) = e_{R_P}(PR_P; C_P) = e_{R_P}(PR_P; R_P) = \operatorname{len}_{R_P}(R_P).$$

For the second equality, it suffices to show that P is analytically unramified; then the equality follows from Theorem 3.4(a).

Since I is generated by monomials in the x_i, the associated prime P has the form $P = (x_{i_1},\ldots,x_{i_j})R$. This is, of course, standard when S is a polynomial ring. Since S is not a polynomial ring, we justify this statement. First note that each ideal $(x_{i_1},\ldots,x_{i_j})R$ is prime because the sequence $\mathbf{x}$ is a regular system of parameters. Since $R = S/I$ is Cohen-Macaulay, the prime P is minimal in $\operatorname{Spec}(R)$. Let $\pi\colon S \to R$ be the canonical surjection, and set $Q = \pi^{-1}(P)$. The prime Q is a minimal prime for any primary decomposition of I, and it follows that Q is a minimal prime for any primary decomposition of the radical $\sqrt{I}$.

Because the sequence $\mathbf{x}$ is regular and contained in the Jacobson radical of S, a result of Heinzer, Mirbagheri, Ratliff, and Shah [10, (4.10)] implies that there are non-negative integers $u, e_{1,1},\ldots,e_{1,n}, e_{2,1},\ldots,e_{2,n},\ldots,e_{u,1},\ldots,e_{u,n}$ such that

$$I = \cap_{s=1}^{u}(x_1^{e_{s,1}},\ldots,x_n^{e_{s,n}})S.$$

Since each ideal $(x_{i_1},\ldots,x_{i_j})S$ is prime, it is straightforward to show that one has $\sqrt{(x_1^{e_{s,1}},\ldots,x_n^{e_{s,n}})S} = (x_{i_1},\ldots,x_{i_j})S$ and hence

$$\sqrt{I} = \cap_{s=1}^{u}(x_1^{\epsilon_{s,1}},\ldots,x_n^{\epsilon_{s,n}})S \tag{3.5.1}$$

where

$$\epsilon_{s,i} = \begin{cases} 0 & \text{if } e_{s,i} = 0 \\ 1 & \text{if } e_{s,i} \neq 0. \end{cases}$$

Since each $(x_1^{\epsilon_{s,1}},\ldots,x_n^{\epsilon_{s,n}})S$ is prime, the intersection (3.5.1) is a primary decomposition. It follows that $P = (x_1^{\epsilon_{s,1}},\ldots,x_n^{\epsilon_{s,n}})S$ for some index s, so P has the desired form.

It follows that $R/P \cong S/(x_{i_1},\ldots,x_{i_j})S$ is a regular local ring. Thus, the completion $\widehat{R/P}$ is also a regular local ring. In particular, the ring $\widehat{R/P}$ is an integral domain, so it is reduced, and P is analytically unramified by definition. □

4. Length and Betti Numbers

This section contains results relating lengths and multiplicities to Betti numbers of semidualizing modules. We begin with a general result for modules of infinite projective dimension.

Lemma 4.1. *Let R be a local ring such that $\operatorname{Ass}(R) = \operatorname{Min}(R)$. Let C be a finitely generated R-module of infinite projective dimension, and consider an exact sequence*

$$R^{a_1} \xrightarrow{\partial} R^{a_0} \to C \to 0.$$

Assume that for each $P \in \operatorname{Ass}(R)$ one has $\operatorname{len}_{R_P}(C_P) \leqslant \operatorname{len}_{R_P}(R_P)$. Then $a_1 \geqslant a_0$.

Proof. Suppose that $a_1 < a_0$, that is, that $a_1 - a_0 + 1 \leqslant 0$. Set $K = \operatorname{Ker}(\partial)$ and consider the exact sequence

$$0 \to K \to R^{a_1} \xrightarrow{\partial} R^{a_0} \to C \to 0.$$

Localize this sequence at an arbitrary $P \in \operatorname{Ass}(R)$, and count lengths to find that

$$0 \leqslant \operatorname{len}_{R_P}(K_P) \leqslant (a_1 - a_0 + 1) \operatorname{len}_{R_P}(R_P) \leqslant 0.$$

It follows that $K_P = 0$ for all $P \in \operatorname{Ass}(R)$.

Set $L = \operatorname{Im}(\partial)$ and localize the exact sequence

$$0 \to K \to R^{a_1} \xrightarrow{\tau} L \to 0$$

to conclude that $L_P \cong R_P^{a_1}$ for each $P \in \operatorname{Ass}(R)$. That is, the R-module L has rank a_1. Hence, the third step in the next sequence:

$$a_1 \geqslant \mu_R(L) \geqslant \operatorname{rank}_R(L) = a_1.$$

The first step is from the surjection τ. It follows that $\mu_R(L) = \operatorname{rank}_R(L)$, hence we conclude that L is free; see, e.g., [15, (1.12)]. The exact sequence

$$0 \to L \to R^{a_0} \to C \to 0$$

implies that $\operatorname{pd}_R(C)$ is finite, a contradiction. So, we have $a_1 \geqslant a_0$, as desired. □

Theorem 4.2. *Let $(R, \mathfrak{m})$ be a local ring such that $\operatorname{Ass}(R) = \operatorname{Min}(R)$. Let C be a semidualizing R-module such that $C \not\cong R$, and consider an exact sequence*

$$R^{a_1} \xrightarrow{\partial} R^{a_0} \to C \to 0.$$

For each $P \in \operatorname{Ass}(R)$, assume that one of the following conditions holds:

(1) *R_P is Gorenstein;*
(2) *$P^2 R_P = 0$;*
(3) *$R_P \cong S/\mathfrak{n}^e$ for some regular local ring $(S, \mathfrak{n})$ containing a field and some integer $e \geqslant 1$; or*
(4) *R_P is isomorphic to a localization of a Cohen-Macaulay ring of the form S/I where S is a regular local ring containing a field with $x_1, \ldots, x_n \in S$ a regular system of parameters for S such that I is generated by monomials in the x_i.*

Then $\operatorname{len}_{R_P}(C_P) = \operatorname{len}_{R_P}(R_P)$ for each $P \in \operatorname{Ass}(R)$. It follows that $a_1 \geqslant a_0$ and that $e(J; C) = e(J; R)$ for each $\mathfrak{m}$-primary ideal J.

Proof. We first show that $\operatorname{len}_{R_P}(C_P) = \operatorname{len}_{R_P}(R_P)$ for each $P \in \operatorname{Ass}(R)$. If P satisfies condition (1) or (2), this is a consequence of Fact 2.7. Under conditions (3) and (4), we apply Lemma 3.1 and Corollary 3.5, respectively.

Now, the conclusion $a_1 \geqslant a_0$ follows from Lemma 4.1, since Fact 2.7 implies that $\operatorname{pd}_R(C) = \infty$. The equality $e(J; C) = e(J; R)$ for each $\mathfrak{m}$-primary ideal J follows from the additivity formula as in the proof of Theorem 3.2. □

The next result shows how the existence of a non-trivial semidualizing module yields an affirmative answer to [12, (2.6)].

Corollary 4.3. *Let R be a Cohen-Macaulay local ring with a dualizing module D. Let C be a semidualizing R-module such that $D \not\cong C \not\cong R$. If for each $P \in \operatorname{Ass}(R)$ one of the conditions (1)–(4) from Theorem 4.2 holds, then $\beta_1^R(D) \geqslant 2\beta_0^R(D)$.*

Proof. Set $C^\dagger = \operatorname{Hom}_R(C,D)$. The condition $D \not\cong C$ implies that $C^\dagger \not\cong R$ by Fact 2.5(f). Hence, Theorem 4.2 implies that $\beta_1^R(C^\dagger) \geqslant \beta_0^R(C^\dagger)$ and $\beta_1^R(C) \geqslant \beta_0^R(C)$. This explains the second step in the next sequence:

$$\beta_1^R(D) = \beta_1^R(C)\beta_0^R(C^\dagger) + \beta_0^R(C)\beta_1^R(C^\dagger) \geqslant 2\beta_0^R(C)\beta_0^R(C^\dagger) = 2\beta_0^R(D).$$

The first and third steps follow from Fact 2.5(g). □

5. Counting Semidualizing Modules

Here we prove Theorem 1.3, starting with the lemma that makes our proof work.

Lemma 5.1. *Let R be a Cohen-Macaulay local ring of codimension 2. Let C be a semidualizing R-module such that $\beta_0^R(C) \leqslant \beta_1^R(C)$. Then C is dualizing for R.*

Proof. Case 1: The ring R is complete. Since R is Cohen-Macaulay and complete, it admits a dualizing module D. Suppose by way of contradiction that $C \not\cong D$. Note that the assumption $\beta_0^R(C) \leqslant \beta_1^R(C)$ implies that $C \not\cong R$. Because of the condition $R \not\cong C$, Fact 2.3 implies that $\beta_0^R(C) \geqslant 2$.

Fact 2.5(b) implies that the R-module $C^\dagger = \operatorname{Hom}_R(C,D)$ is semidualizing. The assumption $C \not\cong D$ implies that $C^\dagger \not\cong R$ by Fact 2.5(f). We conclude from Fact 2.7 that $\operatorname{pd}_R(C^\dagger) = \infty$, and hence $\beta_1^R(C^\dagger) \geqslant 1$. In summary, we have

$$2 \leqslant \beta_0^R(C) \leqslant \beta_1^R(C) \qquad\qquad 1 \leqslant \beta_1^R(C^\dagger).$$

This explains the third step in the following sequence:

$$\begin{aligned} 1 + \beta_0^R(D) = \beta_1^R(D) &= \beta_0^R(C)\beta_1^R(C^\dagger) + \beta_1^R(C)\beta_0^R(C^\dagger) \\ &\geqslant 2 + \beta_0^R(C)\beta_0^R(C^\dagger) = 2 + \beta_0^R(D). \end{aligned}$$

The first step is from [12, (2.7)], and the remaining steps are from Fact 2.5(g). This sequence is impossible, so this completes the proof in Case 1.

Case 2: The general case. Let $\widehat{R}$ denote the completion of R. Suppose by way of contradiction that C is not dualizing for R. The completion $\widehat{C}$ is a semidualizing $\widehat{R}$-module by Fact 2.6. Furthermore, we have

$$\beta_0^{\widehat{R}}(\widehat{C}) = \beta_0^R(C) \leqslant \beta_1^R(C) = \beta_1^{\widehat{R}}(\widehat{C}).$$

Thus, Case 1 implies that the $\widehat{R}$-module $\widehat{C}$ is dualizing for $\widehat{R}$, and it follows from Fact 2.6 that C is dualizing for R. This completes the proof. □

The next result contains Theorem 1.3 from the introduction. Note that we do not assume here that R admits a dualizing module.

Theorem 5.2. *Assume that R is a local Cohen-Macaulay ring of codimension 2. Then R has at most two semidualizing modules up to isomorphism, namely R and a dualizing module (if one exists).*

Proof. Let C be a semidualizing R-module, and assume that $C \not\cong R$. In particular, we have $\operatorname{pd}_R(C) = \infty$ by Fact 2.7. We show that $\beta_0^R(C) \leqslant \beta_1^R(C)$. (Once this is done, Lemma 5.1 implies that C is dualizing for R, as desired.)

To prove that $\beta_0^R(C) \leqslant \beta_1^R(C)$, use Fact 2.6 as in Case 2 of the previous proof to pass to the completion of R in order to assume that R is complete. Cohen's structure theorem implies that there is an epimorphism of local rings $\tau\colon Q \to R$ such that Q is a complete regular local ring such that $\dim(Q) = \operatorname{edim}(R) = \dim(R) + 2$. Set $I = \operatorname{Ker}(\tau)$, and note that $\operatorname{ht}(I) = 2$.

Since Q is regular, we have $\operatorname{pd}_Q(R) < \infty$. Hence, the Auslander-Buchsbaum Formula implies that $\operatorname{pd}_Q(R) = 2$. Because of the condition $\operatorname{ht}(I) = 2$, the Hilbert-Burch Theorem implies that there is an $n \times (n+1)$ matrix $(x_{i,j})$ of elements in Q such that $I = I_n(x_{i,j})$, the ideal generated by the maximal minors of the matrix $(x_{i,j})$.

Let $(X_{i,j})$ be an $n \times (n+1)$ matrix of variables over A, and consider the local ring $A = Q[\{X_{i,j}\}]_{\mathfrak{M}}/I_n(X_{i,j})$ where $\mathfrak{M}$ is generated by the maximal ideal of Q and the variables $X_{i,j}$. The map $\varphi\colon A \to R$ given by $\overline{X_{i,j}} \mapsto \overline{x_{i,j}}$ is a well-defined local ring epimorphism. It is straightforward to show that $\operatorname{Ker}(\varphi)$ is generated by the sequence

$$\overline{X_{i,j}} - \overline{x_{i,j}} \qquad i = 1, \ldots, n \text{ and } j = 1, \ldots, n+1. \tag{5.2.1}$$

This sequence has length $n(n+1)$.

We claim that the generating sequence (5.2.1) is A-regular. Indeed, since Q is Cohen-Macaulay, the ring A is Cohen-Macaulay by [3, (5.17)]. Also, we have

$$\dim(A) = \dim(Q) + n(n+1) - 2 = \dim(R) + n(n+1)$$

by [3, (5.12(a))]. Since R is a quotient of A by the sequence (5.2.1) of length $n(n+1)$, the displayed equalities imply that the sequence (5.2.1) is part of a system of parameters for A. Since A is Cohen-Macaulay, this proves the claim.

To prove that $\beta_0^R(C) \leqslant \beta_1^R(C)$, it suffices to show that $\operatorname{len}_{R_P}(C_P) = \operatorname{len}_{R_P}(R_P)$ for each $P \in \operatorname{Ass}(R)$, by Lemma 4.1. Fix a prime $P \in \operatorname{Ass}(R)$, and consider the induced map $\tau_P\colon A_{\mathfrak{p}} \to R_P$ where $\mathfrak{p} = \varphi^{-1}(P) \in \operatorname{Spec}(A)$. Since τ is surjective with kernel generated by an A-regular sequence, it follows that τ_P is surjective with kernel generated by an $A_{\mathfrak{p}}$-regular sequence. Since Q is an integral domain, the ring $A_{\mathfrak{p}}$ is reduced by [3, (5.8)]. (In fact, the ring $A_{\mathfrak{p}}$ is a normal domain by [3, (6.3)].) In particular, for each $\mathfrak{q} \in \min(A_{\mathfrak{p}})$, the ring $(A_{\mathfrak{p}})_{\mathfrak{q}}$ is Gorenstein. Also, since Q is complete and $A_{\mathfrak{p}}$ is essentially of finite type over Q, we know that $A_{\mathfrak{p}}$ is excellent. It follows that, for each $\mathfrak{q} \in \min(A_{\mathfrak{p}})$, the ring $(A_{\mathfrak{p}})_{\mathfrak{q}}/\mathfrak{q}(A_{\mathfrak{p}})_{\mathfrak{q}} \otimes \widehat{A_{\mathfrak{p}}}$ is Gorenstein. Thus, the conclusion $\operatorname{len}_{R_P}(C_P) = \operatorname{len}_{R_P}(R_P)$ follows from [14, (2.8(b))]. □

Second proof of Theorem 5.2. As in the first proof, we assume that R is complete. With the notation of the first proof, consider the complete local ring $\widehat{A} = Q[\![\{X_{i,j}\}]\!]/I_n(X_{i,j})$. Since R is complete, the map $\widehat{\varphi}\colon \widehat{A} \to R$ is a well-defined local ring epimorphism, and $\operatorname{Ker}(\widehat{\varphi})$ is generated by an $\widehat{A}$-regular sequence. From [14, (4.8)] we know that $\widehat{A}$ has at most two semidualizing modules up to isomorphism, namely $\widehat{A}$ and a dualizing module. Fact 2.6 implies that R has at most two semidualizing modules up to isomorphism, namely R and a dualizing module. □

Remark 5.3. We include two proofs of Theorem 5.2 for the following reasons.

First, the second proof utilizes [14, (4.8)]. The statement of this result is a bit heavy on the notation, and the proof is technical. Second, the first proof is more self-contained, and it showcases the connection between Question 1.4 and the question of characterizing the semidualizing modules over a given ring.

We conclude with the graded version of Theorem 5.2.

Corollary 5.4. *Let $R = \coprod_{i\geqslant 0} R_i$ be a graded Cohen-Macaulay ring of codimension 2 such that R_0 is local. Then R has at most two graded semidualizing modules, up to isomorphism, namely R and a dualizing module.*

Proof. Let C be a graded semidualizing R-module, and suppose that $C \ncong R$. Let $\mathfrak{m}$ be the homogeneous maximal ideal of R. Lemma 2.11 implies that $C_\mathfrak{m} \ncong R_\mathfrak{m}$, so Theorem 5.2 implies that $C_\mathfrak{m}$ is dualizing for $R_\mathfrak{m}$. It follows from Lemma 2.13 that C is dualizing for R, as desired. □

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References

1. M. Auslander, S. Ding, and Ø. Solberg, *Liftings and weak liftings of modules*, J. Algebra **156** (1993), 273–397. MR 94d:16007
2. W. Bruns and J. Herzog, *Cohen-Macaulay rings*, revised ed., Studies in Advanced Mathematics, vol. 39, University Press, Cambridge, 1998. MR 1251956 (95h:13020)
3. W. Bruns and U. Vetter, *Determinantal rings*, Lecture Notes in Mathematics, vol. 1327, Springer-Verlag, Berlin, 1988. MR 953963 (89i:13001)
4. L. W. Christensen, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1839–1883. MR 2002a:13017
5. L. W. Christensen, J. Striuli, and O. Veliche, *Growth in the minimal injective resolution of a local ring*, J. Lond. Math. Soc., to appear, `arXiv:math/0812.4672v1`.
6. A. Frankild and S. Sather-Wagstaff, *Reflexivity and ring homomorphisms of finite flat dimension*, Comm. Algebra **35** (2007), no. 2, 461–500. MR 2294611
7. ———, *The set of semidualizing complexes is a nontrivial metric space*, J. Algebra **308** (2007), no. 1, 124–143. MR 2290914
8. A. A. Gerko, *On suitable modules and G-perfect ideals*, Uspekhi Mat. Nauk **56** (2001), no. 4(340), 141–142, translation in Russian Math. Surveys **56** (2001), no. 4, 749–750. MR 1 861 448
9. R. Hartshorne, *Residues and duality*, Lecture Notes in Mathematics, No. 20, Springer-Verlag, Berlin, 1966. MR 36 #5145
10. W. Heinzer, A. Mirbagheri, L. J. Ratliff, Jr., and K. Shah, *Parametric decomposition of monomial ideals. II*, J. Algebra **187** (1997), no. 1, 120–149. MR 1425562 (97k:13001)
11. B. Herzog, *On the macaulayfication of local rings*, J. Algebra **67** (1980), no. 2, 305–317. MR 0602065 (82c:13029)
12. D. A. Jorgensen and G. J. Leuschke, *On the growth of the Betti sequence of the canonical module*, Math. Z. **256** (2007), no. 3, 647–659. MR 2299575 (2008a:13018)
13. S. Sather-Wagstaff, *Bass numbers and semidualizing complexes*, Fez Conference Proceedings, to appear, `arXiv:math.AC/0801.4743`.
14. ———, *Semidualizing modules and the divisor class group*, Illinois J. Math. **51** (2007), no. 1, 255–285. MR 2346197
15. S. Sather-Wagstaff and D. White, *An Euler characteristic for modules of finite G-dimension*, Math. Scand. **102** (2008), no. 2, 206–230. MR 2437187

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