Available online at www.sciencedirect.com
ScienceDirect
Journal of Algebra 319 (2008) 3120–3127
JOURNAL OF Algebra
www.elsevier.com/locate/jalgebra

The Gale transform and multi-graded determinantal schemes

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Received 23 May 2007

Available online 21 December 2007

Communicated by Steven Dale Cutkosky

Abstract

Eisenbud and Popescu showed that certain finite determinantal subschemes of projective spaces defined by maximal minors of adjoint matrices of homogeneous linear forms are related by Veronese embeddings and a Gale transform. We extend this result to adjoint matrices of multihomogeneous multilinear forms. The subschemes now lie in products of projective spaces and the Veronese embeddings are replaced with Segre embeddings.

Keywords: Gale transform; Veronese and Segre embeddings; Determinantal subschemes

1. Introduction

The Gale transform takes a sufficiently nice set Γ of $r+s+2$ points in the projective space $\mathbf{P}^r$ to a set Γ' of the same number of points in $\mathbf{P}^s$. In [4] and [3] Eisenbud and Popescu give a scheme theoretic definition of the Gale transform for finite Gorenstein schemes. Let Γ be a finite Gorenstein scheme, L a line bundle on Γ, $V \subset H^0(L)$ a subspace, K the canonical line bundle on Γ, and $V^\perp \subset H^0(KL^{-1})$ the annihilator of V under Serre duality. The Gale transform of the linear series (V, L) is defined to be $(V^\perp, KL^{-1})$. In [4, Theorem 6.1] Eisenbud and Popescu

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doi:10.1016/j.jalgebra.2007.11.013

prove the following relation between the Gale transform and certain determinantal schemes. Let ν_d denote the dth Veronsese embedding.

Theorem 1.1. *Let V and W be k-vector spaces of dimension $r+1$ and $s+1$ respectively. Let $\phi : F \to V \otimes W$ be a map of vector spaces with* $\dim_k F = r+s$, *and let*

$$\phi_V : F \otimes k[V] \to W \otimes k[V](1)$$

be the corresponding map of free modules over the polynomial ring $k[V]$. Let ϕ_W be the analogous map over $k[W]$, and let $\Gamma_V \subset \mathbf{P}^r$ and $\Gamma_W \subset \mathbf{P}^s$ be the schemes defined by the ideals of minors $I_{s+1}(\phi_V) \subset k[V]$ and $I_{r+1}(\phi_W) \subset k[W]$, respectively.

If Γ_V and Γ_W are both zero-dimensional then they are both Gorenstein, there is a natural isomorphism between them, and

$$\nu_{s-1}(\Gamma_V) \text{ is the Gale transform of } \nu_{r-1}(\Gamma_W).$$

In this paper we generalize this result to cover linear transformations ϕ from a vector space F to a tensor product of three or more vector spaces. The determinantal schemes will now lie in products of two or more projective spaces and the Veronese embeddings will be replaced by Segre embeddings. For the most part the proof is very similar to that given by Eisenbud and Popescu; however, in one place it differs substantially. We work over the complex numbers because we use some results from [6] and we do not know whether these remain valid for other fields.

2. Preliminaries

Recall the following facts about cohomology of projective space. See for instance Theorem III.5.1 of [7]. Consider the projective space $\mathbf{P} = \mathbf{P}(W)$, where W is a vector space of dimension $s+1$. Let a be an integer. There is at most one integer i for which $H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(a)) \neq 0$. If $a \geqslant 0$, then $H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(a)) = \mathrm{Sym}_a(W)$. If $-1 \geqslant a \geqslant -s$, then $H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(a)) = 0$ all i. If $-s-1 \geqslant a$, then $H^s(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(a)) = \mathrm{Sym}_{-a-s-1}(W)^*$. An analogous result holds for products of projective spaces. This result is also probably well known, but we do not know of a reference.

Proposition 2.1. *Denote the product of k projective spaces $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ by $\mathbf{P}$. Assume $n_j > 0$ for all j. Let $\mathbf{a} = (a_1, \ldots, a_k)$ be a k-tuple of integers. Then there is at most one i for which the cohomology group $H^i(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(\mathbf{a})) \neq 0$. If for any j we have $-1 \geqslant a_j \geqslant -n_j$, then all the cohomology groups vanish. Otherwise, for each j let i_j be the unique integer for which $H^{i_j}(\mathbf{P}^{n_j}, \mathcal{O}(a_j)) \neq 0$. Set $i = i_1 + \cdots + i_k$. Then*

$$H^i\big(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(\mathbf{a})\big) = H^{i_1}\big(\mathbf{P}^{n_1}, \mathcal{O}(a_1)\big) \otimes \cdots \otimes H^{i_k}\big(\mathbf{P}^{n_k}, \mathcal{O}(a_k)\big).$$

Proof. We proceed by induction on k. The case $k = 1$ is the previously mentioned result for a single projective space. For the induction step apply the Leray spectral sequence, see [6, p. 462], to the projection $\pi : \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k} \to \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{k-1}}$. The $E_2^{p,q}$ term is

$$H^p\big(\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{k-1}}, R^q\pi_*\mathcal{O}_{\mathbf{P}}(\mathbf{a})\big).$$

Set $\mathbf{a}' = (a_1, \dots, a_{k-1})$ and $\mathbf{P}' = \mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{k-1}}$. Then

$$R^q \pi_* \mathcal{O}_{\mathbf{P}}(\mathbf{a}) = H^q\big(\mathbf{P}^{n_k}, \mathcal{O}(a_k)\big) \otimes \mathcal{O}_{\mathbf{P}'}(\mathbf{a}'),$$

use [7, Exercise III.8.3]. Since $H^q(\mathbf{P}^{n_k}, \mathcal{O}(a_k))$ is nonzero for at most one q, $E_2^{p,q}$ is nonzero for at most one q. This means that from this point on all the differentials in the spectral sequence must be 0 and the spectral sequence degenerates at step 2.

$$H^p\big(\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_{k-1}}, R^q \pi_* \mathcal{O}_{\mathbf{P}}(\mathbf{a})\big) = H^q\big(\mathbf{P}^{n_k}, \mathcal{O}(a_k)\big) \otimes H^p\big(\mathbf{P}', \mathcal{O}_{\mathbf{P}'}(\mathbf{a}')\big).$$

The induction hypothesis then finishes the proof. □

Next we give a proposition that is an analogue of Proposition 2.5 in [4], which is in turn closely related to Proposition 1.1 and Corollary 1.3 in [3]. The proof is very similar to the proof found in those references.

Proposition 2.2. *Denote the product of k projective spaces* $\mathbf{P}^{n_1} \times \cdots \times \mathbf{P}^{n_k}$ *by* $\mathbf{P}$. *Assume* $n_i > 0$ *for all i. Set* $n = \Sigma n_i$. *Let* $\mathbf{a} = (a_1, \dots, a_k)$ *be a k-tuple of positive integers. Set* $b_i = -a_i - n_i - 1$ *for all i and* $\mathbf{b} = (b_1, \dots, b_k)$. *Let* $\Gamma \subset \mathbf{P}$ *be a finite Gorenstein scheme. Let V be the linear series cut out on* Γ *by* $H^0(\mathbf{P}, \mathcal{O}_{\mathbf{P}}(\mathbf{a}))$. *Then we have that the Gale transform of V is the image of* $\operatorname{Ext}^{n-1}_{\mathbf{P}}(\mathcal{I}_\Gamma, \mathcal{O}_{\mathbf{P}}(\mathbf{b}))$ *in* $H^0(\Gamma, K_\Gamma(-\mathbf{a}))$.

Proof. From the exact sequence

$$0 \to \mathcal{I}_\Gamma \to \mathcal{O}_{\mathbf{P}} \to \mathcal{O}_\Gamma \to 0$$

and Proposition 2.1 we get the sequence

$$\cdots \to H^0\big(\mathcal{O}_{\mathbf{P}}(\mathbf{a})\big) \to H^0\big(\mathcal{O}_\Gamma(\mathbf{a})\big) \to H^1\big(\mathcal{I}_\Gamma(\mathbf{a})\big) \to 0$$

which identifies $V^\perp$ with $H^1(\mathcal{I}_\Gamma(\mathbf{a}))^*$. By Serre duality $\operatorname{Ext}^{n-1}_{\mathbf{P}}(\mathcal{I}_\Gamma, \mathcal{O}_{\mathbf{P}}(\mathbf{b})) \cong H^1(\mathcal{I}_\Gamma(\mathbf{a}))^*$. □

Let U_i be a vector space of dimension $s_i + 1 \geqslant 2$ for $i = 1, \dots, k$, $k \geqslant 2$. Let F be a vector space of dimension $n = \Sigma s_i$. Let $\phi : F \to \bigotimes_{i=1}^k U_i$ be a linear transformation. For each j, $1 \leqslant j \leqslant k$, we have a corresponding map of free modules:

$$\phi_j : F \otimes k\Big[\bigoplus_{i \neq j} U_i\Big] \to U_j \otimes k\Big[\bigoplus_{i \neq j} U_i\Big](1, \dots, 1).$$

We have subschemes $\Gamma_j \subset \prod_{i \neq j} \mathbf{P}^{s_i}$ defined by the ideals of $(s_j + 1) \times (s_j + 1)$ minor determinants $I_{s_j+1}(\phi_j) \subset k[\bigoplus_{i \neq j} U_i]$.

Example 2.3. It is interesting to see the matrices that give the maps ϕ_i once coordinates are fixed. We consider the case $k = 3$. Let $\{f_i\}_{i=1}^n$, $\{x_j\}_{j=1}^{s_1+1}$, $\{y_t\}_{t=1}^{s_2+1}$ and $\{z_l\}_{l=1}^{s_3+1}$ be bases for the vector spaces F, U_1, U_2 and U_3, respectively. Note that $\phi(f_i) = \sum_{j=1}^{s_1+1} \sum_{t=1}^{s_2+1} \sum_{l=1}^{s_3+1} a_{i,j,t,l} x_j \otimes y_t \otimes z_l$ for some scalars $a_{i,j,t,l}$. Thus, the map ϕ is given by the four-dimensional array of numbers

$[a_{i,j,t,l}]$. The map ϕ_1 is the composition of the map $\phi \otimes 1 : F \otimes k[U_2 \oplus U_3] \to U_1 \otimes U_2 \otimes U_3 \otimes k[U_2 \oplus U_3]$ with the map $1 \otimes \mu : U_1 \otimes U_2 \otimes U_3 \otimes k[U_2 \oplus U_3] \to U_1 \otimes k[U_2 \oplus U_3]$ that sends $u_1 \otimes u_2 \otimes u_3 \otimes G$ to $u_1 \otimes u_2 u_3 G$,

$$\begin{aligned}
\phi_1(f_i \otimes G) &= (1 \otimes \mu)\bigl(\phi(f_i) \otimes G\bigr) \\
&= (1 \otimes \mu)\left(\sum_{j=1}^{s_1+1} \sum_{t=1}^{s_2+1} \sum_{l=1}^{s_3+1} a_{i,j,t,l} x_j \otimes y_t \otimes z_l \otimes G\right) \\
&= (1 \otimes \mu)\left(\sum_{j=1}^{s_1+1} x_j \otimes \left(\sum_{t=1}^{s_2+1} \sum_{l=1}^{s_3+1} a_{i,j,t,l} y_t \otimes z_l \otimes G\right)\right) \\
&= \sum_{j=1}^{s_1+1} x_j \otimes \left(\sum_{t=1}^{s_2+1} \sum_{l=1}^{s_3+1} a_{i,j,t,l} y_t z_l G\right).
\end{aligned}$$

Thus, the i, j entry of the matrix representing ϕ_1 is $\sum_{t=1}^{s_2+1} \sum_{l=1}^{s_3+1} a_{i,j,t,l} y_t z_l$. Similarly, the i, t entry of the matrix representing ϕ_2 is $\sum_{j=1}^{s_1+1} \sum_{l=1}^{s_3+1} a_{i,j,t,l} x_j z_l$ and the i, l entry of the matrix representing ϕ_3 is $\sum_{j=1}^{s_1+1} \sum_{t=1}^{s_2+1} a_{i,j,t,l} x_j y_t$. Observe that the matrices for ϕ_1, ϕ_2 and ϕ_3 are obtained by collapsing the four-dimensional array $[a_{i,j,t,l}]$ along two of its dimensions. This pattern holds for arbitrary k.

For notational purposes, for any integer i let $[i]$ denote the $(k-1)$-tuple of integers all of whose entries are i.

Proposition 2.4. *Focus on Γ_j and denote it by Γ. Similarly denote $\prod_{i \neq j} \mathbf{P}^{s_i}$ by $\mathbf{P}$. From the Eagon–Northcott complex we get the following sequence which gives a locally free resolution of $\mathcal{I}_\Gamma$ as a sheaf on $\mathbf{P}$. Similar results hold for the other Γ's,*

$$\begin{aligned}
&0 \to \mathrm{Sym}_{n-s_j-1} U_j^* \otimes \mathcal{O}_{\mathbf{P}}\bigl(-[n]\bigr) \\
&\to \mathrm{Sym}_{n-s_j-2} U_j^* \otimes \bigwedge^{n-1} F \otimes \mathcal{O}_{\mathbf{P}}\bigl(-[n-1]\bigr) \to \cdots \\
&\to U_j^* \otimes \bigwedge^{s_j+2} F \otimes \mathcal{O}_{\mathbf{P}}\bigl(-[s_j+2]\bigr) \\
&\to \bigwedge^{s_j+1} F \otimes \mathcal{O}_{\mathbf{P}}\bigl(-[s_j+1]\bigr) \to \mathcal{I}_\Gamma \to 0.
\end{aligned}$$

Proof. Restricted to each of the standard affine open patches that cover **P** this becomes a standard Eagon–Northcott complex. One checks that the twists are correct by looking at transition data. Alternatively one could proceed as follows. Let S be the homogeneous coordinate ring of **P** embedded in projective space by the linear system of divisors of type $(1, 1, \ldots, 1)$. The map ϕ_j induces a map of S modules $F \otimes S \to U_j \otimes S(1, \ldots, 1)$. Associated to this map we have an Eagon–Northcott complex. What we have here is the associated complex of sheaves. □

For future use we want to compute the degrees of the schemes Γ_j. To do this we use the following special case of Porteous's formula.

Proposition 2.5. *Let X be a purely n-dimensional scheme and let $\mathcal{L}$ be a line bundle on X associated to an effective Cartier divisor H. Let ϕ be a $q \times p$ matrix whose entries are regular sections of $\mathcal{L}$, and let W_r be the subscheme of X defined by the vanishing of the $(r+1) \times (r+1)$ minors of ϕ. Define a class D_r in the Chow group $A_{n-(p-r)(q-r)}(X)$ by*

$$D_r = \prod_{i=0}^{p-r-1} \left[\frac{\binom{q+i}{r}}{\binom{r+i}{r}} \right] H^{(p-r)(q-r)}.$$

(a) *If W_r is nonempty, then each irreducible component of W_r has codimension at most $(p-r)(q-r)$ in X.*
(b) *If $D_r \neq 0$ in $A_{n-(p-r)(q-r)}(X)$ (for which it is enough to check $H^{(p-r)(q-r)} \neq 0$) then W_r is not empty.*
(c) *If the codimension of every irreducible component of W_r in X is exactly $(p-r)(q-r)$ and X is Cohen–Macaulay (meaning locally Cohen–Macaulay) then W_r is Cohen–Macaulay (meaning locally Cohen–Macaulay) and the class of W_r in $A_{n-(p-r)(q-r)}(X)$ is D_r.*

Proof. This is a generalization of Lemma 1.1.1 of [8]. Its proof is an easy application of material in [1, Chapter II, Section 4, subsections (iii) and (iv)], and [5, Theorem 14.4]. □

Proposition 2.6. *Assume that Γ_j is zero-dimensional. Then the degree of Γ_j is*

$$\frac{n!}{\prod_{i=1}^{k} s_i!}.$$

Proof. Apply Proposition 2.5 with $p = s_j + 1$, $q = n$, $r = s_j$ and H a divisor of type $(1, \ldots, 1)$. With this, H^{n-s_j} is the degree of the standard Segre embedding of $\prod_{i \neq j} \mathbf{P}^{s_i}$, which from its Hilbert function is easily calculated. Thus the degree of Γ_j is

$$\frac{n!}{s_j!(n-s_j)!} \times \frac{(n-s_j)!}{\prod_{i \neq j} s_i!}. \qquad \square$$

We will be using a spectral sequence associated to hypercohomology to compute an Ext. Here we briefly describe the spectral sequence we will use. See [6] starting on pages 445 and 705 for details. Given a complex of sheaves $K^0 \to K^1 \to K^2 \to \cdots$ with differential d on a space X there is a spectral sequence that computes the hypercohomology of the complex with $E_2^{p,q}$ term equal to $H_d^q(H^p(X, K^*))$.

3. The main results

Denote $\prod_{i \neq j} \mathbf{P}^{s_i}$ by $\mathbf{P}_j$. For any $(k-1)$-tuple of positive integers $\mathbf{a}$ denote by $\sigma_{\mathbf{a}}(\mathbf{P}_j)$ the Segre embedding given by divisors of type $\mathbf{a}$.

Proposition 3.1. *The following four conditions are equivalent.*

(a) *For all* i, $1 \leqslant i \leqslant k$, Γ_i *is zero-dimensional and locally Gorenstein,* $\operatorname{codim} I_{s_i+1}(\phi_i) = n - s_i$, *and* $\operatorname{codim} I_{s_i}(\phi_i) = n - s_i + 1$.

(b) Γ_1 *and* Γ_2 *are both zero-dimensional.*

(c) $\operatorname{codim} I_{s_1+1}(\phi_1) = n - s_1$, *and* $\operatorname{codim} I_{s_1}(\phi_1) = n - s_1 + 1$.

(d) Γ_1 *is zero-dimensional and locally Gorenstein.*

Proof. This proposition is an analogue of Proposition 6.2 of [4]. The proof is quite similar.

(b) ⇔ (c). By definition Γ_1 is zero-dimensional iff $\operatorname{codim} I_{s_1+1}(\phi_1) = n - s_1$. So we suppose that Γ_1 is zero-dimensional and need to show that Γ_2 is zero-dimensional iff ϕ_1 never drops rank by more than 1. Let

$$\phi_{F,1} : U_1^* \otimes k\left[F^* \oplus \bigoplus_{i=3}^{k} U_i\right] \to U_2 \otimes k\left[F^* \oplus \bigoplus_{i=3}^{k} U_i\right](1, \ldots, 1)$$

be a map of free modules induced by ϕ. For each fixed point $p \in \prod_{i=3}^{k} \mathbf{P}(U_i)$ we obtain a map of free modules (well defined up to a scalar multiple)

$$\phi_{F,p} : U_1^* \otimes k[F^*] \to U_2 \otimes k[F^*](1).$$

For points $p_1 \in \mathbf{P}(U_1)$, $p_2 \in \mathbf{P}(U_2)$, and $p \in \prod_{i=3}^{k} \mathbf{P}(U_i)$ we have that (p_1, p) is in the support of Γ_1 iff the generalized column (see [2, Exercise A2.18]) of $\phi_{F,p}$ corresponding to p_1 has a generalized zero. Similarly, (p_2, p) is in the support of Γ_2 iff the generalized row of $\phi_{F,p}$ corresponding to p_2 has a generalized zero. Some generalized column of $\phi_{F,p}$ has a generalized zero iff some generalized row of $\phi_{F,p}$ has a generalized zero. We are assuming that Γ_1 is zero-dimensional, so there are only finitely many $p \in \prod_{i=3}^{k} \mathbf{P}(U_i)$ such that some generalized row or column of $\phi_{F,p}$ has a generalized zero, and for each such only finitely many generalized columns have generalized zeros. Therefore, Γ_2 is zero-dimensional iff for any such p no generalized column of $\phi_{F,p}$ has two, and hence infinitely many, generalized zeros. In other words Γ_2 is finite iff ϕ_1 never drops rank by more than one.

(c) ⇔ (d). Again Γ_1 is zero-dimensional iff $\operatorname{codim} I_{s_1+1}(\phi_1) = n - s_1$. So we suppose that Γ_1 is zero-dimensional and need to show that Γ_1 is locally Gorenstein iff ϕ_1 never drops rank by more than 1. Since both conditions are local we may localize at each maximal ideal corresponding to one of the points in the support of Γ_1. Let ω be the stalk of the dualizing sheaf of Γ_1 at that point. The resolution in Proposition 2.4 gives that $\omega \cong \operatorname{Sym}_{n-s_1-1} \operatorname{coker} \phi_1$, where the cokernel is also localized at that point. Now ϕ_1 does not drop rank by more than one at that point iff $\operatorname{coker} \phi_1$ is locally principal at that point iff $\operatorname{Sym}_{n-s_1-1} \operatorname{coker} \phi_1$ is locally principal at that point iff ω is locally principal at that point iff Γ_1 is locally Gorenstein at that point.

(a) ⇔ (d). Here we can use that we already know that (b), (c), and (d) are all equivalent. Clearly (a) implies (d) is obvious. For (d) implies (a) first note that since the numbering of the U_i was arbitrary (d) implies (b) gives that (d) implies that all the Γ_i are finite. Having that all the Γ_i are finite (b) implies (c) and (b) implies (d) and the fact that the numbering of the U_i was arbitrary gives the rest of (a). □

Theorem 3.2. *Assume that one and hence all of the equivalent conditions in Proposition* 3.1 *are satisfied.*

(a) *All of the* Γ_i *are naturally isomorphic to each other.*
(b) *Set* $\mathbf{a} = (a_1, \dots, a_{k-1})$ *where* $a_i = n - \sum_{l=i}^{k-1} s_l - 1$. *Assume* $s_k \geqslant 2$ *and if* $k = 2$ *then* $s_1 \geqslant 2$. *Set* $\mathbf{b} = (b_1, \dots, b_{k-2}, b_k)$ *where for* $1 \leqslant i \leqslant k-2$, $b_i = \sum_{l=i+1}^{k-1} s_l$ *and* $b_k = n - s_k - 1$. *Then the Gale transform of* $\sigma_{\mathbf{a}}(\Gamma_k)$ *is* $\sigma_{\mathbf{b}}(\Gamma_{k-1})$.

Proof. (a) The proof of this is very similar to the proof of the analogous part of Theorem 6.1 of [4]. On $\mathbf{P} := \prod_{i=1}^{k} \mathbf{P}(U_i)$ the map ϕ corresponds to a map of sheaves

$$\phi_0 : \mathcal{O}_{\mathbf{P}}(-1, \dots, -1)^n \to \mathcal{O}_{\mathbf{P}}.$$

We define a subscheme $\Gamma \subset \mathbf{P}$ by setting $\mathcal{O}_\Gamma := \operatorname{coker} \phi_0$. Let $p_i : \mathbf{P} \to \mathbf{P}_i$ be projection along the ith factor. We claim that p_i induces an isomorphism from Γ to Γ_i. All the Γ_i are then naturally isomorphic because they are all naturally isomorphic to Γ.

We have $\phi_i = (p_i)_* \phi_0(0, \dots, 0, 1, 0, \dots, 0)$ where the 1 is in the ith spot. Thus $(p_i)_* \mathcal{O}_\Gamma(0, \dots, 0, 1, 0, \dots, 0) = \operatorname{coker} \phi_i$. By hypothesis $I_{s_i}(\phi_i)$ defines the empty set, so $\operatorname{coker} \phi_i$ is a line bundle on Γ_i. In particular $p_i(\Gamma) = \Gamma_i$. By symmetry, the projection of Γ to any other $\mathbf{P}_j$ is Γ_j. Since fibers of p_i project isomorphically to any $\mathbf{P}_j$, $i \neq j$, and Γ_j is zero-dimensional, this shows in particular that the map $(p_i)|_\Gamma : \Gamma \to \Gamma_i$ is a finite map. The fact that the push forward by p_i of a line bundle is a line bundle now implies that $(p_i)|_\Gamma$ is an isomorphism.

(b) Here our proof differs substantially from the proof of the analogous part of Theorem 6.1 of [4]. Knowing that all the Γ_i are naturally isomorphic we may think of $\sigma_{\mathbf{a}}(\Gamma_k)$ and $\sigma_{\mathbf{b}}(\Gamma_{k-1})$ as different embeddings of the same scheme. Set $\mathbf{c} = (c_1, \dots, c_{k-1})$ where $c_i = -a_i - s_i - 1$. From Proposition 2.2 we see that the Gale transform of $\sigma_{\mathbf{a}}(\Gamma_k)$ will be given by the linear series cut out on Γ_k by $\operatorname{Ext}_{\mathbf{P}_k}^{n-s_k-1}(\mathcal{I}_{\Gamma_k}, \mathcal{O}_{\mathbf{P}_k}(\mathbf{c}))$. To compute this we apply $\operatorname{Hom}(., \mathcal{O}_{\mathbf{P_k}}(\mathbf{c}))$ to the free part of the exact sequence in Proposition 2.4 and use the previously mentioned spectral sequence to compute hypercohomology. To start we wish to compute all the cohomology groups of all the sheaves appearing in the complex. Our claim is that only one of these is not zero. In what follows use Proposition 2.1 as needed.

Start with the last sheaf in the complex

$$\begin{aligned} &\operatorname{Hom}\bigl(\operatorname{Sym}_{n-s_k-1} U_k^* \otimes \mathcal{O}_{\mathbf{P}_k}[-n], \mathcal{O}_{\mathbf{P}_k}[c]\bigr) \\ &\quad = \operatorname{Sym}_{n-s_k-1} U_k \otimes \mathcal{O}_{\mathbf{P}_k}(b_1, \dots, b_{k-2}, 0). \end{aligned}$$

This has cohomology group H^0 equal to the linear system that embeds $\sigma_{\mathbf{b}}(\Gamma_{k-1})$ and all other cohomology groups equal to zero. The next to the last sheaf will involve

$$\mathcal{O}_{\mathbf{P}_k}(b_1 - 1, \dots, b_{k-2} - 1, -1)$$

which has all its cohomology groups equal to zero because of the -1 as the right most twist. As we proceed on one step at a time all the twists go down by one at each step. We cannot hope to have a nonzero cohomology group until the right most twist reaches $-s_{k-1} - 1$. At that point the next to the right most twist will be $b_{k-2} - s_{k-1} - 1 = -1$. So now that twist forces all cohomology groups to be zero. Proceeding on, each time a twist gets low enough to allow nonzero cohomology the next twist to the left has reached -1. The lowest the first twist gets is $-s_1 + 1$ which is not low enough to give any nonzero cohomology.

Thus in the spectral sequence the E_2 term has only one nonzero entry. This means that the spectral sequence degenerates at step 2 and E_2 computes the Ext. As we saw the one nonzero entry equals the linear system that embeds $\sigma_{\mathbf{b}}(\Gamma_{k-1})$.

There remains the question of whether $\sigma_{\mathbf{a}}(\Gamma_k)$ and $\sigma_{\mathbf{b}}(\Gamma_{k-1})$ each span their target projective spaces. From very basic facts about Gale transforms this will be the case if the sum of the dimensions of the target projective spaces plus 2 equals the degree of Γ_k. We computed the degree of Γ_k in Proposition 2.6 and the dimensions of the target projective spaces are easily computed. One easily checks the desired equality. □

Remark 3.3.

(1) To make the notation easier the proposition is stated for k and $k-1$. One can obtain many analogous results by renumbering the U_i's.
(2) One obtains the Eisenbud–Popescu result as the case where $k=2$.

Acknowledgments

The authors would like to thank Ragnar-Olaf Buchweitz and Aldo Conca for helpful discussions about [4]. We would also like to thank Anthony Geramita for making the second named author aware of this question. In fact Propositions 2.5 and 2.6 in the case $k=3$ were obtained by Geramita and the second named author in joint work before the two current authors began joint work on the project.

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