

Research Statement – Susan M. Cooper

1. INTRODUCTION & MOTIVATION

My research is motivated by the links between algebraic geometry and commutative algebra¹. I am especially interested in obtaining geometric information of an object in projective space using tools which encode algebraic invariants associated to an ideal defining the object.

After the projective plane was co-ordinatized² (in the 1800's), many people became interested in algebraic invariants of geometric properties of plane curves. A commonly used technique was to produce a function that assigns a number which is independent of the choice of coordinates to a geometric object. David Hilbert's³ [28] work in invariant theory led him to introduce the notions of *Hilbert functions* and *graded Betti numbers* of homogeneous ideals.

Let k be an algebraically closed field of characteristic 0 and let R denote the polynomial ring $k[x_0, \dots, x_n]$. We say that $F \in R$ is *homogeneous* if every term of F has the same degree (e.g. $x_0^2 - 10x_1x_3$ is homogeneous of degree 2). An ideal $I \subseteq R$ is *homogeneous* if it can be generated by a set of homogeneous polynomials. We group the elements of I by degree which results in a collection of finite dimensional vector spaces. For example, let $I = (F, G) \subseteq R = k[x_0, x_1, x_2]$ where $F = x_0 - x_2$ and $G = x_1^2 - x_1x_2$. Any homogeneous polynomial of degree 2 in I has the form $c_1x_0F + c_2x_1F + c_3x_2F + c_4G$, where c_1, c_2, c_3, c_4 are scalars. Thus, the dimension of the *degree two piece* I_2 of I is 4, denoted $\dim_k I_2 = 4$.

Hilbert studied how the dimensions of the space of invariants of degree d varies with d . To record this data, if $I \subseteq R$ is a homogeneous ideal then we incorporate the degree-by-degree dimensions of I in a sequence called the *Hilbert function* of R/I , denoted $H(R/I)$. More precisely, $H(R/I)$ is the sequence $\{H(R/I, d)\}_{d \geq 0}$ of non-negative integers where

$$H(R/I, d) := \dim_k (R/I)_d = \dim_k R_d - \dim_k I_d = \binom{n+d}{d} - \dim_k I_d \text{ for } d \geq 0.$$

For example, if $I = (x_0 - x_2, x_1^2 - x_1x_2) \subseteq R = k[x_0, x_1, x_2]$ then $H(R/I) = (1, 2, 2, 2, \dots)$.

Related to the Hilbert function of a homogeneous ideal $I \subseteq R$ are invariants obtained by looking at the relations on the generators of I , and the relations on these relations (called the syzygies), etc. When we study these relations we find an exact sequence

$$0 \rightarrow \bigoplus_i R(-i)^{\beta_{i,i}(I)} \rightarrow \bigoplus_i R(-i)^{\beta_{i-1,i}(I)} \rightarrow \dots \rightarrow \bigoplus_i R(-i)^{\beta_{0,i}(I)} \rightarrow I \rightarrow 0,$$

where $R(-i)$ is the ring R but with a shift in the grading (i.e., $[R(-i)]_t = R_{i+t}$) and the maps are given by matrices of homogeneous polynomials of positive degree. The numbers $\beta_{j,i}(I)$ are the *graded Betti numbers for I* and are invariants associated to I .

Question: What information about an ideal is encoded in its Hilbert function and graded Betti numbers?

My research programs have been driven by this type of question, but with a geometric flavor. In particular, my work currently focuses on

- (1) Hilbert functions and graded Betti numbers:
 - (a) Finding geometric consequences of extremal behavior of Hilbert functions of points;
 - (b) Investigating the Eisenbud-Green-Harris Conjecture for ideals of points;
 - (c) Studying properties of fat point subschemes;
- (2) Behavior of determinantal schemes.

These topics not only have applications in algebraic geometry and commutative algebra, but there are related open problems in coding theory, graph theory, computational complexity and statistics. I think that these connections lead to exciting topics for graduate and undergraduate research. I enthusiastically look forward to being a mentor for student research.

¹MSC 13D40, 13P99, 14C99, 14M06, 14M12

²Projective n -space, $\mathbb{P}^n(k)$, is the set of $(n+1)$ -tuples (s_0, \dots, s_n) of elements of a field k modulo the equivalence relation $(s_0, \dots, s_n) \sim (ts_0, \dots, ts_n)$ for $t \neq 0$ in k . A nice exposition of the history of the development of \mathbb{P}^n can be found in [13].

³Hilbert's 14th Problem in his famous speech to the International Congress of Mathematicians in 1900 concerned finding a finite basis for invariants in more general settings.

2. HILBERT FUNCTIONS

2.1. Projective Space and Macaulay's Theorem.

A 0-dimensional scheme is a finite set of points in \mathbb{P}^n to each point of which is attached a 0-dimensional ring (i.e., an Artinian ring). The field has a deep history and yet still manages to host many exciting open problems [34]. The importance of 0-dimensional schemes was demonstrated when Castelnuovo showed that non-trivial data about a curve embedded in projective space can be retrieved from the finite sets of points arising as the general hyperplane sections of the curve [16]. Many algebraic ideas have been introduced in order to obtain information about points. The Hilbert function has played a central role in many problems. To give further details, we let $R := k[x_0, \dots, x_n]$ where k is an algebraically closed field and consider homogeneous ideals $I \subseteq R$.

Hilbert functions have been extensively studied. Perhaps the most celebrated result in the algebraic setting is *Macaulay's Theorem* which implies that $H(R/I)$ can be described using *lex ideals*.

Definition 2.1.1. A *lex ideal* is a monomial ideal L which is minimally generated by monomials $\{m_1, \dots, m_r\}$, where, for $j = 1, \dots, r$, all monomials of degree $\deg(m_j)$ which are larger than m_j in the degree-lexicographic ordering are contained in L .

Macaulay's Theorem [31, 39] Let $I \subseteq R$ be a homogeneous ideal. Then there exists a lex ideal L such that $H(R/I) = H(R/L)$.

To demonstrate, let $I = (x_0^2 + x_0x_1, x_0^3x_1, x_0^5 + x_1^5) \subseteq R = k[x_0, x_1]$. Then $\dim_k I_1 = 0, \dim_k I_2 = 1, \dim_k I_3 = 2, \dim_k I_4 = 4, \dim_k I_i = \dim_k R_i$ for $i \geq 5$. Let $L = L_1 + L_2 + \dots$ be the lex ideal where L_i is generated by the largest $\dim_k I_i$ monomials in the degree-lexicographic ordering; so $L_1 = (0), L_2 = (x_0^2), L_3 = (x_0^3, x_0^2x_1), L_4 = (x_0^4, x_0^3x_1, x_0^2x_1^2, x_0x_1^3)$, etc. Then $H(R/L) = H(R/I) = (1, 2, 2, 2, 1, 0, \dots)$.

Many people have tried to generalize Macaulay's Theorem to families of homogeneous ideals with special properties. Indeed, these generalizations have developed in a variety of different directions (for some examples see [1, 21, 29, 30, 35]). On the geometric side, Geramita-Maroscia-Roberts [24] considered Hilbert functions associated to ideals of points in \mathbb{P}^n . More precisely, if \mathbb{X} is a set of points and $\mathbf{I}(\mathbb{X}) \subseteq R$ is the ideal consisting of all the homogeneous polynomials vanishing on \mathbb{X} , then the *Hilbert function of \mathbb{X}* is $H(\mathbb{X}) := H(R/\mathbf{I}(\mathbb{X}))$. Geramita-Maroscia-Roberts [24] characterize the sequences which arise as Hilbert functions of finite sets of distinct, reduced points in \mathbb{P}^n . Empowered with such a characterization, the Hilbert function of a set of points \mathbb{X} can then be exploited to obtain both algebraic data about $\mathbf{I}(\mathbb{X})$ and geometric information about \mathbb{X} . For example, E. D. Davis [11] and Bigatti-Geramita-Migliore [2] use extremal behavior of Hilbert functions of points to guarantee that some subsets have special properties.

2.2. Points and the Eisenbud-Green-Harris Conjecture.

The Eisenbud-Green-Harris Conjecture is an attempt to generalize Macaulay's Theorem which has gained much recent attention. It concerns ideals containing regular sequences. Technically speaking, $\{F_1, \dots, F_n\}$ in $S := k[x_1, \dots, x_n]$ is a *regular sequence of forms* if each F_i is a homogeneous polynomial such that F_{i+1} is not a zero-divisor on $S/(F_1, \dots, F_i)S$ for each i . For example, if $2 \leq a_1 \leq a_2 \leq \dots \leq a_n$ are integers then $\{x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n}\}$ is a regular sequence. Such sequences of monomials are key elements in the analog of the lex ideal from Macaulay's Theorem.

Definition 2.2.1. Let $\mathbb{A} := \{a_1, \dots, a_n\}$ where $2 \leq a_1 \leq a_2 \leq \dots \leq a_n$ are integers. A *lex-plus-powers ideal with respect to \mathbb{A}* is a monomial ideal in S of the form $J + (x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$ where J is a lex ideal.

For example, let $\mathbb{A} = \{2, 3, 3\}$. Then $L_1 = (x_1^2, x_2^3, x_3^3, x_1x_2^2, x_1x_2x_3)$ is a lex-plus-powers ideal with respect to \mathbb{A} , but $L_2 = (x_1^2, x_2^3, x_3^3, x_1x_2^2, x_1x_2x_3, x_2^2x_3)$ is not since $x_1x_2^3 >_{d\text{-lex}} x_2^2x_3$ and $x_1x_2^3 \notin L_2$.

Fix $\mathbb{A} = \{a_1, \dots, a_n\}$ where $2 \leq a_1 \leq a_2 \leq \dots \leq a_n$ are integers. Clements-Lindström [4] show (in a combinatorial fashion) that for any *monomial ideal* $M \subseteq S$ containing $\{x_1^{a_1}, \dots, x_n^{a_n}\}$ there is a lex-plus-powers ideal L with respect to \mathbb{A} such that $H(S/M) = H(S/L)$. Cooper-Roberts have generalized this:

Lemma 2.2.2. [5, 10] *If $I \subseteq S$ is any homogeneous ideal containing $\{x_1^{a_1}, \dots, x_n^{a_n}\}$, then there is a lex-plus-powers ideal L with respect to \mathbb{A} such that $H(S/I) = H(S/L)$.*

Eisenbud-Green-Harris were interested in homogeneous ideals containing a regular sequence that is not equal to the special sequence $\{x_1^{a_1}, \dots, x_n^{a_n}\}$. They conjectured the following:

Eisenbud-Green-Harris (EGH) Conjecture [3, 14, 15, 20] If $I \subseteq S$ is a homogeneous ideal containing a regular sequence F_1, \dots, F_n of forms of degrees $\deg(F_i) = a_i$, then there is a lex-plus-powers ideal L with respect to \mathbb{A} such that $H(S/I) = H(S/L)$.

The work of Greene-Kleitman [26] has been applied to obtain degree-by-degree bounds on the Hilbert functions of homogeneous ideals containing $\{x_1^{a_1}, \dots, x_n^{a_n}\}$ (see [5, 10]). Using these bounds, I proved:

Theorem 2.2.3. [6] *The EGH Conjecture is true in the case when $I \subseteq S = k[x_1, x_2, x_3]$ has minimal generators which are all in the same degree such that two of the minimal generators form a regular sequence in $k[x_1, x_2]$.*

Apart from Theorem 2.2.3, and despite much effort of many researchers, the EGH Conjecture is known to be true only in some exceptional cases. The conjecture has been proven in the cases where L is an *almost complete intersection* [19], and when $n = 2$ [5, 36]. Mermin-Peeva-Stillman [32] have results for ideals containing squares of the variables. Most recently, Caviglia-Maclagan [3] have proven that the EGH Conjecture is true if $a_j > \sum_{i=1}^{j-1} (a_i - 1)$ for $j = 1, \dots, n$. On the geometric side, I [5, 7] have studied the conjecture for ideals of reduced, distinct, finite point sets (see Theorem 2.2.6).

A *complete intersection* is a set of points $\mathbb{Y} \subseteq \mathbb{P}^n$ such that $\mathbf{I}(\mathbb{Y})$ is generated by n homogeneous polynomials which form a regular sequence. The list $\{d_1, \dots, d_n\}$ of the degrees of the generating homogeneous polynomials is the *type* of the complete intersection. A finite set \mathbb{X} of distinct, reduced points is contained in some (in fact, many) complete intersections, if we have the freedom to pick the degrees of the defining polynomials. A main idea that has motivated my research is to fix these degrees and find conditions that a given Hilbert function must satisfy in order for it to be the Hilbert function of a subset of points in a complete intersection.

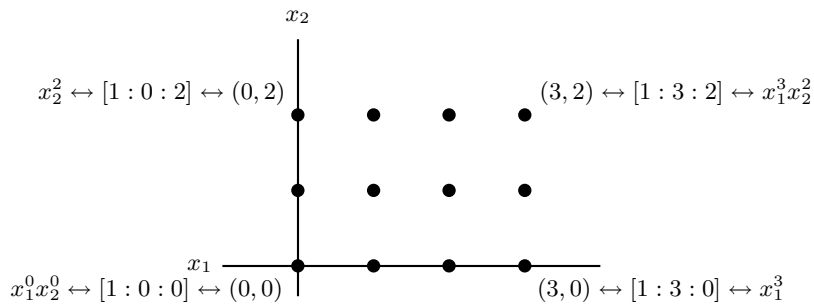
Question 2.2.4. Fix integers $2 \leq d_1 \leq d_2 \leq \dots \leq d_n$ and let \mathcal{H} be the Hilbert function of some finite set of distinct points in \mathbb{P}^n . Do there exist finite sets of distinct, reduced points \mathbb{X} and \mathbb{Y} such that: (1) $\mathbb{X} \subseteq \mathbb{Y}$; (2) $H(\mathbb{X}) = \mathcal{H}$; and (3) \mathbb{Y} is a complete intersection of type $\{d_1, \dots, d_n\}$?

Question 2.2.4 has been answered for complete intersections of the form

$$\{[1 : a_1 : \dots : a_n] \mid a_1, \dots, a_n \in \mathbb{Z}, 0 \leq a_1 \leq d_n - 1, 0 \leq a_2 \leq d_{n-1} - 1, \dots, 0 \leq a_n \leq d_1 - 1\} \subseteq \mathbb{P}^n.$$

This set is called the *rectangular complete intersection of type $\{d_1, \dots, d_n\}$* , denoted $\text{RectCI}(d_1, \dots, d_n)$.

Again let $R := k[x_0, x_1, \dots, x_n]$ and $S := k[x_1, \dots, x_n]$. Observe that the points of $\text{RectCI}(d_1, \dots, d_n)$ are in 1-1 correspondence with the monomials of $S/(x_1^{d_1}, x_2^{d_2}, \dots, x_n^{d_n})$. For example, $\text{RectCI}(3,4)$ can be visualized as the following 12 dots (\bullet) below with the corresponding bijections noted:



Moreover, if $\mathbb{X} \subseteq \text{RectCI}(d_1, \dots, d_n)$ then $R/(\mathbf{I}(\mathbb{X}), x_0) = S/J$ where J a monomial ideal containing $x_1^{d_1}, \dots, x_n^{d_n}$. The above mentioned bounds of Greene-Kleitman [26] can then be applied to characterize the Hilbert functions of such quotients S/J . We have the immediate question:

Question 2.2.5. Is the characterization of Hilbert functions of subsets of non-rectangular complete intersections completely controlled by the situation prevailing in the rectangular case? That is, does $\{\mathcal{H} \mid \mathcal{H} = H(\mathbb{X}), \mathbb{X} \subseteq \mathbb{Y} \in \text{CI}(d_1, \dots, d_n)\} = \{\mathcal{H}' \mid \mathcal{H}' = H(\mathbb{W}), \mathbb{W} \subseteq \text{RectCI}(d_1, \dots, d_n)\}$?

In [5, 7] I show that Question 2.2.5 is equivalent to the EGH Conjecture restricted to ideals of points.

Theorem 2.2.6. [5, 7] *The answer to Question 2.2.5 is “yes” in the cases:*

- (1) $n = 2$
- (2) $n = 3$ in the situations: (a) $d_1 = 2$; (b) $d_1 = 3$; (c) $4 \leq d_1 \leq d_2 \leq d_3$ and $d_3 \geq d_1 + d_2 - 1$.

Interestingly enough, although independently discovered, the assumptions on the degrees of the regular sequences needed for (2c) of Theorem 2.2.6 are the same as that of Caviglia-Maclagan [3]. Note, however, that cases (2a) and (2b) are not completely covered by their results.

There is still much work to be done on the EGH Conjecture. We now list two questions which guide some of my current and future programs.

Question 2.2.7. The Eisenbud-Green-Harris Conjecture was originally stated in the geometric setting in the case when the degrees of the regular sequence elements are all 2. Can the techniques I used for Theorem 2.2.6 be generalized to prove the conjecture in this important case for \mathbb{P}^n ?

Question 2.2.8. Essential parts of the proofs for Theorem 2.2.6 rely on algebraic and geometric consequences of when Hilbert functions exhibit extremal behavior as described by Macaulay’s Theorem [2]. Restricting to subsets of complete intersections, what are the algebraic and geometric consequences for Hilbert functions exhibiting extremal behavior as described by Greene-Kleitman [26]?

2.3. Fat Point Subschemes.

It is natural to want a “Macaulay-type” theorem for non-reduced schemes. I am particularly interested in *fat points*.

It is straightforward to verify that if $\mathbb{X} = \{P_1, \dots, P_r\} \subseteq \mathbb{P}^n$ is a finite set of distinct, reduced points, then $\mathbf{I}(\mathbb{X})$ is the intersection of r prime ideals, namely $\mathbf{I}(\mathbb{X}) = \mathbf{I}(P_1) \cap \mathbf{I}(P_2) \cap \dots \cap \mathbf{I}(P_r)$. A *fat point subscheme* \mathbb{Y} of \mathbb{P}^n is defined by a homogeneous ideal $I \subseteq R = k[x_0, \dots, x_n]$ of the form

$$I = \mathbf{I}(P_1)^{m_1} \cap \mathbf{I}(P_2)^{m_2} \cap \dots \cap \mathbf{I}(P_r)^{m_r},$$

where m_1, \dots, m_r are non-negative integers. We call $\mathbb{X} = \{P_1, \dots, P_r\}$ the *support* of \mathbb{Y} and m_1, \dots, m_r the multiplicities of \mathbb{Y} . We denote \mathbb{Y} by the formal sum $m_1P_1 + \dots + m_rP_r$; if each $m_i = m$, then we simply write $\mathbb{Y} = m\mathbb{X}$.

Question 2.3.1. Given non-negative integers m_1, \dots, m_r , is there a characterization of the sequences that occur as Hilbert functions of fat points in \mathbb{P}^n whose multiplicities are m_1, \dots, m_r ?

A major obstacle to answering this question is that the Hilbert function of a fat point scheme depends on both the multiplicities and the arrangement of the support points. Indeed, Question 2.3.1 is challenging and is unknown even in \mathbb{P}^2 with $m_1 = \dots = m_r = 2$. However, if the number of support points is 8 or less then we can write down all of the possible Hilbert functions for any given set of multiplicities [22, 27]. In addition, Geramita-Migliore-Sabourin [25] consider the fat points $2\mathbb{X}$ for sets \mathbb{X} which are in special arrangements.

In 2007 I was awarded an AWM-NSF Mentoring Travel Grant. As part of this grant a collaboration was initiated between myself, B. Harbourne (University of Nebraska - Lincoln) and Z. Teitler (Texas A & M University). Our work is ongoing and is motivated by the following question:

Question 2.3.2. Let $\mathbb{Y} = m_1P_1 + m_2P_2 + \dots + m_rP_r \subseteq \mathbb{P}^2$. What can we say about $H(\mathbb{Y})$ based solely on the multiplicities m_i and geometric information about the support of \mathbb{Y} ?

Our approach is to study $H(\mathbb{Y})$ given the multiplicities and information about which subsets of the points P_1, \dots, P_r are collinear. We define a *reduction vector* d based on the collinearity information and use this vector to give upper and lower bounds on $H(\mathbb{Y})$ in each degree. Moreover, we give an explicit criterion for when these bounds coincide (see Definition 2.3.7 and Theorem 2.3.8). We now describe how to find the reduction vector d .

Definition 2.3.3. Let $\mathbb{Y} = m_1P_1 + m_2P_2 + \dots + m_rP_r \subseteq \mathbb{P}^2$. Let \mathbb{L} be a line defined by the linear form F . The subscheme

$$\mathbb{Y}' := b_1P_1 + b_2P_2 + \dots + b_rP_r,$$

where $b_i = m_i$ if $P_i \notin \mathbb{L}$ and $b_i = \max(m_i - 1, 0)$ if $P_i \in \mathbb{L}$ is the *subscheme of \mathbb{Y} residual to \mathbb{L}* . We write $\mathbb{Y}' := \mathbb{Y} : \mathbb{L}$. (Note that \mathbb{L} does not have to pass through any point of \mathbb{Y} . Moreover, in order to obtain \mathbb{Y}' all we need to know are the multiplicities m_i and which of the points P_i line on \mathbb{L} .)

If Bézout’s Theorem⁴ forces the line \mathbb{L} to be a component of any curve defined by $H \in \mathbf{I}(\mathbb{Y})_t$, then $\dim_k \mathbf{I}(\mathbb{Y})_t = \dim_k \mathbf{I}(\mathbb{Y}'_{t-1})$. However, in general we can only obtain upper and lower bounds on $\dim_k \mathbf{I}(\mathbb{Y})_t$ from $\dim_k \mathbf{I}(\mathbb{Y}')_{t-1}$.

⁴*Bézout’s Theorem* says that if C and C' are two plane curves of degrees d and d' , respectively, which have no common components, then C and C' intersect in dd' points counted with multiplicity.

The reduction vector d is obtained by repeatedly considering residual subschemes of \mathbb{Y} .

Definition 2.3.4. Let $\mathbb{Y} = m_1P_1 + m_2P_2 + \cdots + m_rP_r \subseteq \mathbb{P}^2$. Let $\mathbb{L}_1, \mathbb{L}_2, \dots, \mathbb{L}_n$ be a sequence of lines in \mathbb{P}^2 , not necessarily distinct.

(a) We define fat point schemes $\mathbb{Y}_0, \mathbb{Y}_1, \dots, \mathbb{Y}_n$ by

- $\mathbb{Y}_n := \mathbb{Y}$
- $\mathbb{Y}_{j-1} := \mathbb{Y}_j : \mathbb{L}_j$

We say that $\mathbb{L}_1, \dots, \mathbb{L}_n$ *totally reduces* \mathbb{Y} if $\mathbb{Y}_0 = \emptyset$.

(b) We also define the vector $d = (d_1, d_2, \dots, d_n)$ such that $d_k := \sum \{a_{k,i} : P_i \in \mathbb{L}_k\}$ where

$$\mathbb{Y}_k = a_{k,1}P_1 + a_{k,2}P_2 + \cdots + a_{k,r}P_r.$$

The vector d is the *reduction vector* for \mathbb{Y} induced by $\mathbb{L}_1, \dots, \mathbb{L}_n$.

Example 2.3.5. Let $\mathbb{X} = \{P_1, P_2, P_3\} \subseteq \mathbb{P}^2$ be 3 non-collinear points where P_1 is on the line \mathbb{L}_1 and P_2, P_3 are on the line \mathbb{L}_2 . We study the fat point scheme $\mathbb{Y} = 2P_1 + 3P_2 + 3P_3$.

Consider the sequence of lines $\mathbb{L}_1, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_2, \mathbb{L}_2$. Then we obtain the following fat point subschemes:

- $\mathbb{Y}_5 = \mathbb{Y}$
- $\mathbb{Y}_4 = \mathbb{Y}_5 : \mathbb{L}_2 = 2P_1 + 2P_2 + 2P_3$
- $\mathbb{Y}_3 = \mathbb{Y}_4 : \mathbb{L}_2 = 2P_1 + P_2 + P_3$
- $\mathbb{Y}_2 = \mathbb{Y}_3 : \mathbb{L}_2 = 2P_1$
- $\mathbb{Y}_1 = \mathbb{Y}_2 : \mathbb{L}_1 = P_1$
- $\mathbb{Y}_0 = \mathbb{Y}_1 : \mathbb{L}_1 = \emptyset$

We see that $\mathbb{L}_1, \mathbb{L}_1, \mathbb{L}_2, \mathbb{L}_2, \mathbb{L}_2$ totally reduces \mathbb{Y} with reduction vector $d = (1, 2, 2, 4, 6)$.

Using the entries of a reduction vector d , we define two sequences f_d and F_d . In many cases these sequences give us lower and upper bounds (degree-by-degree) for the Hilbert function of a fat point scheme in \mathbb{P}^2 .

Theorem 2.3.6. [9] *Let $\mathbb{Y} = m_1P_1 + m_2P_2 + \cdots + m_rP_r \subseteq \mathbb{P}^2$. If $\mathbb{L}_1, \dots, \mathbb{L}_n$ is a totally reducing sequence of lines for \mathbb{Y} with associated reduction vector d , then*

$$f_d \leq H(\mathbb{Y}) \leq F_d.$$

The sequences f_d and F_d are straightforward to compute. Moreover, there is an easy to check combinatorial condition which d must satisfy for f_d to equal F_d , giving us an exact value of the Hilbert function in this case. More precisely,

Definition 2.3.7. A vector $v := (v_1, \dots, v_r)$ of non-negative integers is *Bézout* if

- $v_1 \leq v_2 \leq \cdots \leq v_r$ and
- Between any two zero entries of $\Delta v := (v_1, v_2 - v_1, \dots, v_r - v_{r-1})$ there is an entry which is strictly greater than 1.

For example, $v = (1, 3, 3, 5, 6, 7, 9, 10, 10, 10)$ is not Bézout ($\Delta v = (1, 2, 0, 2, 1, 1, 2, 1, 0, 0)$) and $v = (1, 3, 3, 5, 6, 7, 9, 10, 10, 12, 12)$ is Bézout ($\Delta v = (1, 2, 0, 2, 1, 1, 2, 1, 0, 2, 0)$).

Theorem 2.3.8. [9] *Let $\mathbb{Y} = m_1P_1 + m_2P_2 + \cdots + m_rP_r \subseteq \mathbb{P}^2$. If $\mathbb{L}_1, \dots, \mathbb{L}_n$ is a totally reducing sequence of lines for \mathbb{Y} with associated reduction vector d such that d is Bézout, then*

$$f_d = H(\mathbb{Y}) = F_d.$$

In addition to finding $H(\mathbb{Y})$ exactly, when d is Bézout, we have found lower and upper bounds for the graded Betti numbers of $\mathbf{I}(\mathbb{Y})$. These bounds are frequently equal and we give a simple combinatorial formula for when they coincide.

We expect to submit the manuscript [9] by the end of January 2009. The results provide a nice step forward in the characterization of Hilbert functions of fat point schemes. The work naturally leads to the following questions to consider for future projects.

Question 2.3.9. Suppose $\mathbb{Y} = m_1P_1 + \cdots + m_rP_r$ is a fat point scheme in \mathbb{P}^2 with an associated reduction vector d .

- How sharp are the bounds f_d and F_d ?

- When the reduction vector d is Bézout we have $f_d = F_d$ which gives rise to a nice formula for $H(\mathbb{Y})$. If d is not Bézout then we still have the above bounds. When can we nonetheless find a formula for $H(\mathbb{Y})$?
- Is there a geometric meaning for when d is Bézout?
- Are there better bounds if we consider residuation with respect to higher degree curves?

3. ADDITIONAL CURRENT AND FUTURE RESEARCH INTERESTS

Without going into a detailed exposition, we briefly discuss some of my additional current and future research programs.

(1) It is known that given any Hilbert function \mathcal{H} of a reduced zero-dimensional subscheme of \mathbb{P}^n , there is a k -configuration (a reduced subscheme of \mathbb{P}^n whose points are in special arrangement) whose Hilbert function is \mathcal{H} [23]. Thus it is natural to try to classify the Hilbert functions of the fat points whose support is a k -configuration. Indeed this motivates [25]. Sabourin [37, 38] generalizes k -configurations by using complete intersections rather than lines and hyperplanes to build up the reduced points.

Question 3.1.1. What are the Hilbert functions of fat point schemes whose support is the union of complete intersections?

(2) One tool that I used in proving some cases of the EGH Conjecture in my Ph.D. dissertation [5] is the behavior of Hilbert functions under *liaison*. In the 0-dimensional setting, if \mathbb{X} is a subset of a complete intersection \mathbb{Y} then [3, 12, 33] give a formula relating $H(\mathbb{X})$, $H(\mathbb{Y})$ and $H(\mathbb{Y} - \mathbb{X})$.

Question 3.1.2. Is there an analog for this formula for fat points?

(3) *The Gale Transform & Determinantal Schemes:* The Gale transform takes a sufficiently nice set Γ of $r + s + 2$ points in the projective space \mathbb{P}^r to a set Γ' of the same number of points in \mathbb{P}^s . In [17, 18] Eisenbud and Popescu give a scheme theoretic definition of the Gale transform for finite Gorenstein schemes. Eisenbud and Popescu [18] showed that certain finite determinantal subschemes of projective spaces defined by maximal minors of adjoint matrices of homogeneous polynomials of degree 1 are related by Veronese embeddings and a Gale transform. The result is technical to state. In [8] S. Diaz (Syracuse University) and I extended this result to adjoint matrices of multihomogeneous multilinear forms. Our subschemes lie in products of projective spaces and the Veronese embeddings are replaced with Segre embeddings. We are currently investigating the situations involving minors which are not maximal.

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