Quiz 9 Solutions

Please write your solutions to the following exercises in the space provided. You should write legibly and fully explain your work.

Good Luck!

(1) A diagonalization of the matrix \(A\) is given in the form \(P^{-1}AP = D\). List the eigenvalues of \(A\), their algebraic and geometric multiplicities, and bases for the corresponding eigenspaces. [10 pts]

\[
\begin{bmatrix}
-1/4 & 3/4 & -1/4 \\
1/8 & 1/8 & 1/8 \\
\end{bmatrix}
\begin{bmatrix}
1 & 3 & 3 \\
2 & 0 & 2 \\
3 & 3 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 3 & 1 \\
1 & 2 & 0 \\
-1 & 3 & -1
\end{bmatrix}
= 
\begin{bmatrix}
-2 & 0 & 0 \\
0 & 6 & 0 \\
0 & 0 & -2
\end{bmatrix}.
\]

Solution: The eigenvalues of \(A\) are the diagonal entries of \(D\). Thus, \(A\) has eigenvalues \(\lambda_1 = -2\) and \(\lambda_2 = 6\).

Bases for the eigenspaces can be found by selecting the columns of \(P\) which correspond to the diagonal entries of \(D\). So, a basis for \(E_{-2}\) is

\[
\left\{ \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}
\]

and a basis for \(E_6\) is

\[
\left\{ \begin{bmatrix} 3 \\ 2 \\ 3 \end{bmatrix} \right\}.
\]

We see that \(\lambda_1 = -2\) has algebraic and geometric multiplicity 2, and \(\lambda_2 = 6\) has algebraic and geometric multiplicity 1.
(2) Let \( A = \begin{bmatrix} 1 & 3 \\ 2 & 2 \end{bmatrix} \) and \( B = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \). Show that \( A \) and \( B \) are not similar matrices.

**Solution:** If \( A \) and \( B \) were similar, then they would have equal characteristic polynomials.

The characteristic polynomial of \( A \) is

\[
\det \begin{bmatrix} (1 - \lambda) & 3 \\ 2 & (2 - \lambda) \end{bmatrix} = (1 - \lambda)(2 - \lambda) - 6 = \lambda^2 - 3\lambda - 4.
\]

The characteristic polynomial of \( B \) is

\[
\det \begin{bmatrix} (1 - \lambda) & 1 \\ 3 & (-1 - \lambda) \end{bmatrix} = (1 - \lambda)(-1 - \lambda) - 3 = \lambda^2 - 4.
\]

Since \( A \) and \( B \) have different characteristic polynomials they cannot be similar.

(3) Is the following statement true or false? Carefully justify your answer. [4 pts]

Let \( A \) be a diagonalizable \( n \times n \) matrix such that each eigenvalue \( \lambda \) satisfies \( |\lambda| < 1 \). Then \( \lim_{k \to \infty} A^k \) equals the \( n \times n \) zero matrix.

**Solution:** This statement is true. Since \( A \) is diagonalizable, there exists an invertible matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \). We know that the diagonal entries of \( D \) are the eigenvalues of \( A \). Moreover,

\[
\lim_{k \to \infty} A^k = \lim_{k \to \infty} (PDP^{-1})^k = P(\lim_{k \to \infty} D^k)P^{-1}.
\]

Now, \( D^k \) is a diagonal matrix for which each diagonal entry equals the \( k \)th power of an eigenvalue of \( A \). But, as \( k \to \infty \), the \( k \)th power of any eigenvalue of \( A \) approaches 0. That is,

\[
\lim_{k \to \infty} D^k
\]

equals the zero matrix. Therefore, \( \lim_{k \to \infty} A^k \) equals the \( n \times n \) zero matrix.