Quiz 13 Solutions

Please write your solutions to the following exercises in the space provided. You should write legibly and fully explain your work.

Good Luck!

(1) Let $V$ be the vector space $V = \{p(x) \in \mathcal{P}_2 : p(1) = 0\}$.

(a) Find a basis for $V$. Be sure to show all of your work. \[8 \text{ pts}\]

Solution: Note that

$$V = \{p(x) \in \mathcal{P}_2 : p(1) = 0\}$$

$$= \{p(x) = a + bx + cx^2 \in \mathcal{P}_2 : p(1) = 0\}$$

$$= \{p(x) = a + bx + cx^2 \in \mathcal{P}_2 : a + b + c = 0\}$$

$$= \{p(x) = a + bx + (-a - b)x^2 \in \mathcal{P}_2\}$$

$$= \{p(x) = a(1 - x^2) + b(x - x^2) \in \mathcal{P}_2\}$$

$$= \text{span}(1 - x^2, x - x^2)$$

We now verify that our spanning polynomials are also linearly independent. So, suppose we have scalars $c_1$ and $c_2$ such that

$$c_1(1 - x^2) + c_2(x - x^2) = 0 + 0x + 0x^2.$$ 

Equivalently,

$$c_1 + c_2x + (-c_1 - c_2)x^2 = 0 + 0x + 0x^2.$$ 

Comparing coefficients, we see that $c_1 = c_2 = 0$. Thus,

$$\mathcal{B} = \{1 - x^2, x - x^2\}$$

is a basis for $V$.

(b) What is the dimension of $V$? \[2 \text{ pts}\]

Solution: Since the above basis for $V$ has two polynomials, $\dim(V) = 2$. 

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(2) In $M_{22}$, let $B$ be the standard basis $B = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ and let $C$ be the basis $C = \{A, B, C, D\}$, where

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}, \quad D = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$ 

Find the change of basis matrix $P_{C \leftarrow B}$. (Hint: You can find the appropriate coordinate vectors by inspection, but show your work and not just the final matrix.) [10 pts]

**Solution:** We have the linear combinations:

$$E_{11} = A + 0B + 0C + 0D$$
$$E_{12} = -A + B + 0C + 0D$$
$$E_{21} = 0A - B + C + 0D$$
$$E_{22} = 0A + 0B - C + D$$

So,

$$P_{C \leftarrow B} = \begin{bmatrix} [E_{11}]_C & [E_{12}]_C & [E_{21}]_C & [E_{22}]_C \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$