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1 January 16, 2009 (Speaker: Brian Harbourne)

Line bundles are now often equated to locally free sheaves of modules of rank 1. In older work, line bundles sometimes are taken to refer to something essentially *dual* to locally free sheaves of rank 1 (see Exercise II.V.18 of Hartshorne's book, *Algebraic Geometry*).

Let's first look at line bundles in the context of affine varieties.

Let $X = \text{Spec}(R)$ where R is a Noetherian domain (e.g., let k be a field and let $R = k[X_1, \dots, X_n]$ be a polynomial ring in n variables over k). Fundamentally, in this situation, a sheaf of modules is the same thing as a module. More precisely, a **sheaf of modules** \mathcal{F} on X is an assignment of a module $\mathcal{F}(X_f)$ for each $f \in R \setminus \{0\}$; the assignment is required to satisfy a compatibility condition. [In fact, \mathcal{F} is a contravariant functor from the Zariski topology on X (morphisms are inclusions) to a category of modules. There is a bijective correspondence between these sheaves and R -modules:

$$\mathcal{F} \longleftrightarrow \mathcal{F}(X_1)$$

where $X_1 = X$.]

The point here: on affine varieties, sheaves have no added value beyond what ordinary modules already have. Sheaves become important for non-affine varieties (projective varieties and others).

Here we see how to recover \mathcal{F} from $M = \mathcal{F}(X)$. Let M be an R -module. Let \tilde{M} denote the following sheaf \mathcal{F} : let $s \in R \setminus \{0\}$. Then $\mathcal{F}(X_s) = S^{-1}M$ where $R_s = S^{-1}R$ and $S^{-1}M$ is an R_s -module for $S = \{1, s, s^2, \dots\}$. Here we may consider $R_s = R[\frac{1}{s}]$. Note that $S^{-1}M = M \otimes_R R_s$. If \mathcal{F} is a sheaf and $M = \mathcal{F}(X_1)$, then $\mathcal{F} \cong \tilde{M}$.

A **finitely generated locally free sheaf \mathcal{F} of finite rank** (of modules on $\text{Spec}(R)$) is a sheaf where there exist elements $s_1, \dots, s_r \in R$ that generate the unit ideal of R such that $\mathcal{F}(X_{s_i})$ is a free R_{s_i} -module of finite rank. Topologically, X has a finite open cover such that the modules the sheaf assigns to the open sets in the cover are free.

A finitely generated module M on R is **locally free of finite rank** if there are $s_1, \dots, s_r \in R$ that generate the unit ideal of R such that

$S_i^{-1}M = M_{s_i}$ is a free R_{s_i} module of finite rank. We say that a module M is **locally free of rank 1** if the s_i can be chosen such that $M_{s_i} \cong R_{s_i}$ for each i .

Recall that a module M over R is a **projective module** if there exists an R -module N such that $M \oplus N$ is a free R -module.

For the next result, given a prime ideal $P \subset R$, recall that M_P denotes $S^{-1}M$, where $S = R \setminus P$.

Theorem. Let R be a Noetherian domain, $X = \text{Spec}(R)$, M an R -module, and $\mathcal{F} = \tilde{M}$. Then the following are equivalent:

- (i) M is projective and finitely generated.
- (ii) M is locally free of finite rank.
- (iii) M is finitely generated and M_P is an R_P module for all $P \in \text{Spec}(R)$.
- (iv) \mathcal{F} is locally free of finite rank.

Proof. See Exercises 4.11 and 4.12 of Chapter 4.6 of Eisenbud's book, *Commutative Algebra with a View Toward Algebraic Geometry*, hereafter CAVTAG, or, (for written out proofs), Chapter 5 of Artin's *Commutative Algebra* notes. □

Homework.

- (1) Define $f(t) = t^2 - 1$, $g(t) = t^3 - 1$ in the polynomial ring $R = k[t]$. Consider $I = (f(t), g(t)) \subseteq R$. We obtain an exact sequence

$$0 \rightarrow \ker \rightarrow fR \oplus gR \rightarrow I \rightarrow 0.$$

Show that $fR \oplus gR \cong \ker \oplus I$ (i.e., show that I is projective). Show directly (by finding appropriate s_i) that I is locally free.

- (2) Do problem (1), except now let $R = k[t^2, t^3]$.

2 January 23, 2009 (Speaker: Brian Harbourne)

Homework Solutions.

(1) For $R = k[t]$, we see that I is not only locally free but also free since $I = (t - 1)$ is principal.

(2) For $R = k[t^2, t^3]$, we see that I is not free, but it is locally free. We will show that I is locally principal. Letting $s_1 = t^2 - 1$ and $s_2 = t^2$, we have $R_{s_i} = S_i^{-1}R = R_{[\frac{1}{s_i}]}$. Then IR_{s_1} contains the unit $t^2 - 1$, and therefore $IR_{s_1} = R_{s_1}$. Hence IR_{s_1} is free on R_{s_1} .

Considering IR_{s_2} , notice that $(t^3 - 1) - (t^2 - 1) = t^2(t - 1)$ where t^2 is a unit, so $t - 1 \in IR_{s_2}$. But $t^3 - 1 = (t - 1)(t^2 + t + 1)$ and $t^2 - 1 = (t - 1)(t + 1)$, and therefore $(t - 1)R_{s_2} = IR_{s_2}$, so IR_{s_2} is principal in R_{s_2} .

Moreover, $(s_1, s_2) = R$ since $s_2 - s_1 = 1$. Therefore I is locally free.

Note that in both examples, I is a locally principal R -submodule of K , where $K = R_{(0)}$, the field of fractions of R .

Aside. For nice schemes, we can consider a scheme to be an object which is locally isomorphic to the prime spectrum of a ring (compare to n dimensional manifolds, which are topological spaces locally isomorphic to \mathbf{R}^n). In particular, for schemes whose local behavior is the spectrum of a Noetherian domain, called integral Noetherian schemes, a line bundle is up to isomorphism a locally free subsheaf of the constant sheaf of K , where K is the field of rational functions on the scheme.

Let R be a Noetherian domain. A locally principal R -submodule of $K = R_{(0)}$ is known as an **invertible ideal**.

Definition. An R -submodule M of $K = R_{(0)}$ is called a **fractional ideal** if there exists $a \in R \setminus \{0\}$ such that $aM \subseteq R$.

Example. Ideals of R are fractional ideals (with $a = 1$).

Example. Any finitely generated R -submodule $M \subseteq K$ is a fractional ideal, isomorphic to an actual ideal. Given $M = (m_1, \dots, m_r)R \subseteq K$, find a common denominator for the m_i and denote it d . Then $dM \subseteq R$, $dM \cong M$ since R is a domain and is therefore torsion free. Moreover, a submodule of

a ring is an ideal.

Definition. An **invertible ideal** M is a submodule of K such that there exists a submodule $N \subseteq K$ with $MN = R$.

Remark. Invertible ideals are finitely generated.

Proof. Suppose $MN = R$. Then for some $r \in \mathbb{N}$ and $1 \leq i \leq r$ there exist $m_i \in M, n_i \in N$ such that $\sum m_i n_i = 1$. Therefore for all $m \in M$, we have

$$m = m \cdot 1 = \sum m m_i n_i = \sum m_i m n_i$$

where $m n_i \in MN = R$ for each i . Thus $M = (m_1, \dots, m_r)$. □

Corollary. Invertible ideals are fractional ideals, but the converse is not true in general.

Main Fact (Proof in CAVTAG).

- (1) Invertible ideals are locally free of rank 1.
- (2) Any locally free rank 1 R -module is isomorphic to an invertible ideal.

Corollary. There is a bijection between locally free rank 1 R -modules and invertible ideals, and hence line bundles.

The set of invertible ideals in R forms a group denoted $C(R)$, called the group of Cartier divisors:

Let $M, N \in C(R)$. Then MN is an invertible ideal (multiplication occurs in K), and M has inverse $M^{-1} = \{a \in K \mid aM \subseteq R\}$.

Given $f \in K^*$, fR is an invertible ideal and a principal module. In fact $fR = gR$ if and only if there exists $a \in R^*$ such that $f = ag$. So there exists an exact sequence of groups

$$\begin{array}{ccccccc}
 1 & \longrightarrow & R^* & \longrightarrow & K^* & \longrightarrow & C(R) & \longrightarrow & \text{Pic}(R) & \longrightarrow & 1 \\
 & & & & & & \uparrow \subseteq & & & & \\
 & & & & & & K^*/R^* & & & &
 \end{array}$$

where K^*/R^* is the subgroup of principal invertible ideals and $\text{Pic}(R)$ is the Picard group. Generally, the Picard group is defined to be the group of

isomorphism classes of locally free rank 1 R -modules.

Let X be a scheme. Then $\text{Pic}(X)$ is the group of isomorphism classes of line bundles on X . When $X = \text{Spec}(R)$, from the point of view of algebra one can write $\text{Pic}(R)$, but from the point of view of algebraic geometry, it is written $\text{Pic}(\text{Spec}(R))$.

Aside. Let X be a scheme, \mathcal{O}_X the structure sheaf. (For example, if $X = \text{Spec}(R)$, then $\mathcal{O}_X = \tilde{R}$, which is a sheaf of rings.) Assume X is a Noetherian integral scheme. Let \hat{K} be the constant sheaf where K is the field of rational functions on X . Let \hat{K}^* be the constant sheaf for K^* . Then we have the exact sequence of sheaves

$$1 \longrightarrow \mathcal{O}_X^* \longrightarrow \hat{K}^* \longrightarrow \hat{K}^*/\mathcal{O}_X^* \longrightarrow 1 ,$$

giving a long exact sequence in cohomology

$$0 \rightarrow H^0(\mathcal{O}_X^*) \rightarrow H^0(\hat{K}^*) \rightarrow H^0(\hat{K}^*/\mathcal{O}_X^*) \rightarrow H^1(\mathcal{O}_X^*) = \text{Pic}(X) \rightarrow H^1(\hat{K}^*)$$

where $H^1(\hat{K}^*) = 0$ since \hat{K}^* is flasque if X is integral, and where $H^1(\mathcal{O}_X^*) = \text{Pic}(X)$.

3 February 6, 2009 (Speaker: Brian Harbourne)

Let R be a Noetherian locally factorial domain (i.e., (0) is a prime ideal and R_P is a unique factorization domain for every $P \in \text{Spec}(R)$). Let $K = R_{(0)}$.

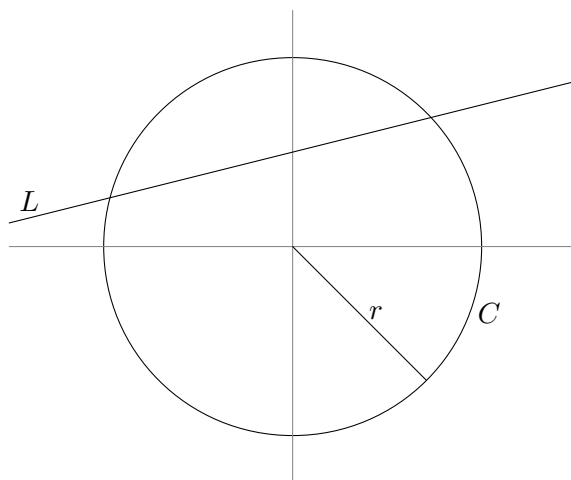
3.1 Various Notions of Divisors.

Intuitively, a divisor in $\mathbb{A}^n = X$ or any space X is a closed subset locally defined by a single equation. Examples include a curve in the plane, a surface in 3-dimensional space, a 4-dimensional surface in 5-dimensional space, and so on.

Definition. A **Weil divisor** on $\text{Spec}(R)$ is an element of the free abelian group generated by the height 1 prime ideals of R .

Example. In the ring $R = \mathbb{C}[x, y]$, a height 1 prime in R is principal, generated by an irreducible polynomial. A Weil divisor here is an integer linear combination of irreducible curves (See Figure 1).

Figure 1: Irreducible curves C and L , given by $x^2 + y^2 - r^2$ and $y - x - 2$.



Define $D_W = 2C - 3L$. Considering the coefficients as integers, we have a single rational function defining the Weil divisor (see below for the definition

of $\text{div}_W(f)$:

$$D_W = \text{div}_W \left(\frac{(x^2 + y^2 - r^2)^2}{(y - x - 2)^3} \right).$$

Definition. A **Cartier divisor** on $\text{Spec}(R)$ is specified by a collection of pairs of polynomials

$$\{(f_i, g_i)\}_{i=1}^r$$

such that $f_i \in K^*$, $g_i \in R$, and the f_i, g_i satisfy the compatibility criterion

$$f_i R_{g_i g_j} = f_j R_{g_j g_i}. \quad (1)$$

Intuitively, f_i defines the Cartier divisor D_C on the open set $\text{Spec}(R_{g_i})$, which is the open subset of $\text{Spec}(R)$ where $g_i \neq 0$. Equation (1) ensures that f_i and f_j define the same divisor on $\text{Spec}(R_{g_i}) \cap \text{Spec}(R_{g_j})$. (The data specifying a Cartier divisor in fact define a subsheaf of the constant sheaf K on $\text{Spec}(R)$, where K is the field of fractions of R . Two different specifications give the same Cartier divisor if they define the same subsheaf.)

Weil divisors and Cartier divisors specify line bundles. Given a Weil divisor $\sum_{i=1}^r m_i P_i$, we get a fractional ideal by taking the product $I = \prod_{i=1}^r P_i^{-m_i}$ where $m > 0$ implies $P^{-m} = \{f \in K \mid fP^m \subseteq R\}$. In fact, this is an invertible ideal since height 1 primes in a UFD are principal. Since I is locally principal and R is a Noetherian locally factorial domain, I is an invertible ideal.

Given an invertible ideal M of R , (i.e., a locally principal submodule M of k), we can obtain a Cartier divisor. Since M is locally principal, there exist elements $(g_1, \dots, g_r) = (1)$ with $g_i \in R$ for all $1 \leq i \leq r$ such that $Mg_i = f_i^{-1}R$ for some $f_i \in K^*$. This gives the Cartier divisor D_C specified by $\{(f_i, g_i)\}_{i=1}^r$.

To obtain a Weil divisor from a Cartier divisor D_C specified by $\{(f_i, g_i)\}_{i=1}^r$, define

$$D_W = \sum_{\substack{P \in \text{Spec}(R) \\ \text{ht}(P)=1}} \mathcal{V}_P(D_C)P$$

where we define $\mathcal{V}_P(D_C)$ as follows. Given P , pick a $g_i \notin P$; such a g_i exists since $(g_1, \dots, g_r) = (1)$. Now R_P is a UFD of Krull dimension 1 (i.e., a discrete valuation ring), and hence R_P is a principal ideal domain (Atiyah-Macdonald, Proposition 9.2). Therefore $PR_P = tR_P$ for some

$t \in k^*$ ($P \neq (0)$ since $\text{ht}(P) = 1$). Now $f_i \in K^*$ so $f_i = \frac{a}{b}$ for some $a, b \in R_P$, and $f_i = \frac{a}{b} = \frac{u_1 t^{n_a}}{u_2 t^{n_b}}$. Define $\mathcal{V}_P(D_C) = n_a - n_b$.

Remark.

- (i) $\mathcal{V}_P(D_C)$ is well-defined because of Equation (1).
- (ii) $\mathcal{V}_P(D_C) = 0$ for all but finitely many P by Krull's *Hauptidealsatz*.

3.2 Global Sections of Line Bundles.

Definition. Let R be a Noetherian locally factorial domain. Let \mathcal{L} be a line bundle on $\text{Spec}(R)$, associated to some Weil divisor $D_W = \sum_{P \in \text{Spec}(R)} m_P P$. Let $f \in K^*$. Define the **principal divisor associated to f** to be $\text{div}_W(f) = \sum_{P \in \text{Spec}(R)} \mathcal{V}_P(f) P$.

From the Weil point of view, the global sections for the line bundle \mathcal{L} are defined to be

$$\Gamma(\mathcal{L}) := \{h \in K^* \mid \mathcal{V}_P(h) + m_P \geq 0 \text{ for all } P \in \text{Spec}(R)\} \cup \{0\}.$$

Remark. $\Gamma(\mathcal{L})$ is an R -module.

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Definition. (Effective Divisor.)

- (i) We say a Weil divisor $D_W = \sum m_P P$ is **effective** if $m_P > 0$ for all P .
- (ii) We say a Cartier divisor D_C represented by $\{(f_i, g_i)\}$ is **effective** if $f_i \in R_{g_i}$ (rather than the field of fractions; this captures the “nonnegative” idea).
- (iii) Let $\mathcal{L}(D_W)$ be the line bundle associated to the Weil divisor. The **space of global sections** of $\mathcal{L}(D_W)$ is defined to be

$$\Gamma(\mathcal{L}(D_W)) := \{f \in K^* \mid \text{div}(f) + D_W \text{ is effective}\} \cup \{0\}.$$

For $\mathcal{L}(D_C)$, we define

$$\Gamma(\mathcal{L}(D_C)) := \{f \in K \mid f \in f_i^{-1} R_{g_i} \forall i\}.$$

Note that $\Gamma(\mathcal{L}(D_W)) = \Gamma(\mathcal{L}(D_C)) = \bigcap f_i^{-1} R_{g_i} = I$ for corresponding divisors D_W , D_C , and the corresponding invertible ideal I .

Example. Let $R = \mathbb{Z}$. Then $D_W = -6(2) + 5(3) = \text{div}\left(\frac{3^5}{2^6}\right) = \left\{\left(\frac{3^5}{2^6}, 1\right)\right\}$ (this follows from Hartshorne Prop II.6.2: in a Noetherian UFD, all divisors are principal). We have $I = \left(\frac{2^6}{3^5}\right)\mathbb{Z} \subseteq \mathbb{Q}$, and $I = \Gamma(\mathcal{L}(D_W)) = \Gamma(\mathcal{L}(D_C)) = \{f \in \mathbb{Q}^* \mid \text{div}(f) + D_W \text{ is effective}\} \cup \{0\}$. But $D_W = \text{div}\left(\frac{3^5}{2^6}\right)$ so $\text{div}(f) + D_W = \text{div}(f) + \text{div}\left(\frac{3^5}{2^6}\right) = \text{div}\left(f \cdot \frac{3^5}{2^6}\right)$. To be effective, the prime factorization of $f \cdot \frac{3^5}{2^6}$ can have no negative exponents. This implies that $f = n \cdot \frac{2^6}{3^5}$ for some integer $n \neq 0$; i.e., $I = \left(\frac{2^6}{3^5}\right)\mathbb{Z}$.

Example. Non-affine examples. Let X be a scheme, \mathcal{L} be a line bundle, and \mathcal{O}_X the structure sheaf (i.e., the line bundle which gives the ring structure of X locally). Note that $\Gamma(\mathcal{L})$ is a $\Gamma(\mathcal{O}_X)$ -module. If X is $\text{Spec}(R)$ for some ring R , then $\Gamma(\mathcal{O}_X) = R$. Often, $\Gamma(\mathcal{O}_X)$ is a field so the global sections $\Gamma(\mathcal{L})$ of \mathcal{L} end up being vector spaces over $\Gamma(\mathcal{O}_X)$.

Let $X = \mathbb{P}_{\mathbb{C}}^1$. In this case X doesn't have a coordinate ring (i.e., there is no ring R with $X = \text{Spec}(R)$), but X has a *homogeneous coordinate ring* $R = \bigoplus_{i \in \mathbb{Z}} R_i$, where R_i is the span (as a complex vector space) of the

homogeneous polynomials in $\mathbb{C}[x, y]$ of degree i . When $i < 0$, $R_i = 0$. We can write $R = \mathbb{C} \oplus \langle x, y \rangle \oplus \langle x^2, xy, y^2 \rangle \oplus \cdots$ and $X = \text{Proj}(R)$, where

$$\begin{aligned} \text{Proj}(R) &= \{P \text{ is a homogeneous prime of } R \mid P \subsetneq (x, y)\} \\ &= \{0\} \cup \underbrace{\{(ax + by) \mid a \text{ or } b \text{ nonzero, } a, b \in \mathbb{C}\}}_{\text{height 1 primes, the closed points of } \mathbb{P}_{\mathbb{C}}^1}. \end{aligned}$$

For $0 \leq t \in \mathbb{Z}$, if $f \in R_t \setminus \{0\}$, then $R_f = R[\frac{1}{f}]$ is graded and $\deg(\frac{1}{f}) = -t$. So $R_f = \bigoplus_{i \in \mathbb{Z}} (R_f)_i$. Let X_f denote the open set in $\mathbb{P}_{\mathbb{C}}^1 = \text{Proj}(R)$ where $f \neq 0$ (i.e., X_f consists of primes not containing f). Then $\mathcal{O}_X(X_f) = (R_f)_0 = \{\frac{g}{f^n} \in C(x, y) \mid \deg(g) = n \deg(f)\} =$ degree 0 elements of R_f .

Consider a Weil divisor $D_W = \sum_{i=1}^r m_i P_i$ with $m_i \in \mathbb{Z}$ and $P_i = (F_i)$ (so the F_i are linear homogeneous polynomials). Let $K(\mathbb{P}_{\mathbb{C}}^1)$ denote the **function field** of $\mathbb{P}_{\mathbb{C}}^1$, which consists of all degree 0 elements of $\mathbb{C}(x, y)$. Then $f \in K^*$ factors as $f = \prod_{j=1}^r H_j^{m_j}$ where each H_j is an irreducible homogeneous polynomial (in this case, linear). Then $\text{div}(f) = \sum_{j=1}^r m_j (H_j)$.

Definition. Define $\deg(D_W) = \sum_{i=1}^r m_i \deg(P_i)$ (in our case, $\deg(P_i) = 1$). In particular, $\deg(\text{div}(f)) = 0$, since $f \in K(\mathbb{P}_{\mathbb{C}}^1)$ implies $\deg f = 0$.

Homework. Compute the global sections $\Gamma(\mathcal{L}(D_W))$ for $D_W = 2(x) + (y)$.

5 February 20, 2009 (Speaker: Brian Harbourne)

Let $X = \mathbb{P}_{\mathbb{C}}^1$, $K = \{f \in \mathbb{C}(x, y) \mid \deg f = 0 \text{ or } f = 0\}$. Recall that given a Weil divisor $D_W = \sum m_i P_i$, where $m_i \geq 0$ and $P_i = (F_i)$ for some linear forms F_i , the global sections are

$$\begin{aligned} \Gamma(\mathcal{L}(D_W)) &= \{\operatorname{div}(f) + D_W \text{ is effective}\} \cup \{0\} \\ &= \left\{ f \in K^\times \mid f \cdot \prod F_i^{m_i} \in R \right\} \\ &= \frac{1}{\prod F_i^{m_i}} R_d \end{aligned}$$

for $d = \deg D_W = \sum m_i$ (since the F_i are linear forms).

Example.

(1) If $\mathcal{O}_X = \mathcal{L}(0)$, then

$$\Gamma(\mathcal{O}_X) = \{f \in K \mid f \cdot 1 \in R\} = R_0 = \mathbb{C}$$

under the grading; that is, $\Gamma(\mathcal{O}_X)$ is a one-dimensional vector space.

(2) Suppose $D_W = 2(x) + (y)$. Then

$$\begin{aligned} \Gamma(\mathcal{L}(D_W)) &= \left\{ f \in K \mid f \cdot x^2 y \in R \right\} \\ &= \left\{ \frac{G}{x^2 y} \mid G \in R_3 \right\} \\ &= \frac{1}{x^2 y} R_3. \end{aligned}$$

Now $\Gamma(\mathcal{O}_X) = \mathbb{C}$ is a field, and $\dim_{\Gamma(\mathcal{O}_X)} \Gamma(\mathcal{L}(D_W)) = 4$ since $R_3 = \langle x^3, x^2 y, x y^2, y^3 \rangle$ (in general, $\dim_{\mathbb{C}} R_d = \binom{d+n}{n}$).

(3) Suppose $D_W = -2(x) + (y)$. Then $\Gamma(\mathcal{L}(D_W)) = \left\{ f \in K \mid f \cdot \frac{y}{x^2} \in R \right\} = \{0\}$ because $\deg f \cdot \deg \frac{y}{x^2} = -\deg f$, and the only polynomial in R with negative degree is 0.

Remark. If $\deg D_W < 0$, then $\Gamma(\mathcal{L}(D_W)) = \{0\}$.

Theorem. Let D_W be any divisor on $\mathbb{P}_{\mathbb{C}}^1$. Let $d = \deg D_W$. Then

$$\dim_{\mathbb{C}} \Gamma(\mathcal{L}(D_W)) = \begin{cases} 0, & \text{if } d < 0 \\ d + 1 = \binom{d+1}{1} = \dim_{\mathbb{C}} R_d & \text{if } d \geq 0 \end{cases}.$$

5.1 Riemann-Roch Theorem for Plane Curves

Let $R = \mathbb{C}[x, y, z] = \mathbb{C}[\mathbb{P}^2]$, the homogeneous coordinate ring of \mathbb{P}^2 . We have

$$\begin{aligned} \mathbb{P}^2 &= \text{Proj}(R) \\ &= \{\text{homogeneous primes } P \subsetneq (x, y, z)\} \\ &= \{(0)\} \cup \underbrace{\{(F) \mid F \in R \text{ homogeneous, irreducible}\}}_{\substack{\text{Height 1 homogeneous primes;} \\ \text{the closure of these are curves.}}} \\ &\quad \cup \underbrace{\{P \mid P \text{ is a homogeneous prime and } \text{ht}(P) = 2\}}_{\text{Closed points of } \mathbb{P}^2}. \end{aligned}$$

In fact, there is a bijective correspondence between

$$\mathbb{C}^3 \setminus \{0\} \ni (a, b, c) \longleftrightarrow P = (ay - bx, az = cy, cy - bz)$$

where P corresponds to the ideal generated by determinants of the maximal minors of the matrix

$$\begin{pmatrix} a & b & c \\ x & y & z \end{pmatrix}.$$

Definition. Let $C \subsetneq \mathbb{P}^2$ be a curve. I.e., C is defined by a nonconstant homogeneous polynomial F . We say C is **smooth** if $\nabla F \neq 0$ at every point of C . Recall that $\nabla F = (\partial F/\partial x, \partial F/\partial y, \partial F/\partial z)$.

Example. The cusp defined by $F(x, y, z) = zy^2 - x^3 = 0$ is not smooth; $\nabla F = (-3x^2, 2yz, y^2)$. If $\nabla F = 0$, then $x = y = 0$ necessarily, which corresponds to a point on the cusp (see Figure 2).

Example. The curve defined by $C : xy - z^2 = 0$ is smooth since $\nabla F = (y, x, -2z) = 0$ if and only if $x = y = z = 0$, which is not a point on the hyperbola in projective space (see Figure 3).

Remark. A smooth plane curve is irreducible.

Definition. Let C be a smooth plane curve of degree d . Then the **genus** g_C of the curve is given by $g_C = \frac{(d-1)(d-2)}{2}$.

Theorem. The curve C is homeomorphic to a torus of genus g_C .

Theorem. (Riemann's Theorem.) Let C be a smooth plane curve of genus g_C . Let D_W be a Weil divisor on C . Then

$$\dim_{\mathbb{C}} \Gamma(\mathcal{L}(D_W)) \geq \deg(D_W) + 1 - g_C.$$

Figure 2: The cusp has a singularity.

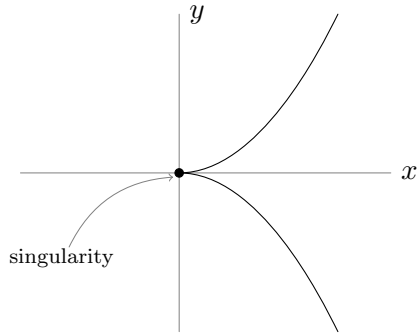
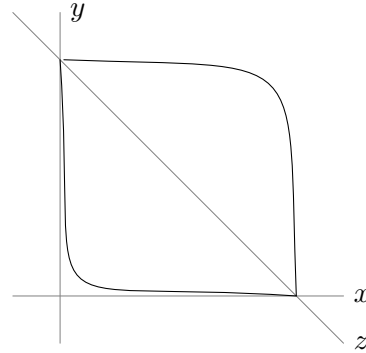


Figure 3: A projective hyperbola.



Roch added the correction term later. This involves the canonical class K_C of the curve (which plays the role that the dualizing module plays in commutative algebra).

Definition. The **canonical class** K_C can be taken to be the Weil divisor given by intersecting C with $d - 3$ general lines in \mathbb{P}^2 (where $d = \deg C$). Different choices of lines give different Weil divisors K_C , but they are all linearly equivalent (i.e., they give isomorphic line bundles).

Example. We can find a Weil divisor on the quartic curve C (i.e., $\deg C = 4$) by intersecting it with one general line in \mathbb{P}^2 (See Figure 4)

Exercise. The Euler characteristic is equal to $2g_C - 2 = \deg K_C = (d - 3)d$.

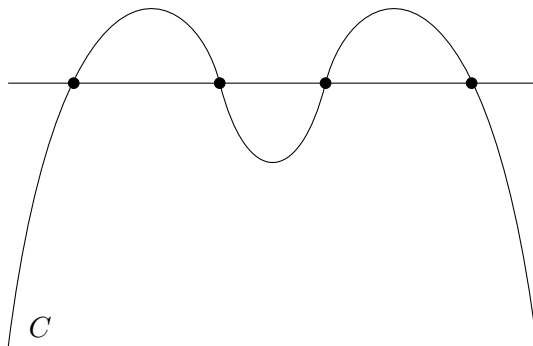
Theorem. (Riemann-Roch for plane curves.) Let C be a smooth plane curve of degree d . Let D_W be a Weil divisor on C . Then

$$\dim_{\mathbb{C}} \Gamma(\mathcal{L}(D_W)) - \dim_{\mathbb{C}} \Gamma(\mathcal{L}(K_C - D_W)) = \deg D_W + 1 - g_C.$$

The modern formulation of Riemann-Roch has us consider Γ as a functor. It becomes the 0-th cohomology functor H^0 , and we get derived functors H^i , $i > 0$. Then

$$\underbrace{\dim H^0(C, \mathcal{L}(D_W))}_{=\Gamma \mathcal{L}(D_W)} - \dim H^1(C, \mathcal{L}(D_W)) = \deg D_W + 1 - g_C.$$

Figure 4: The Weil divisor on C corresponds to the four points in \mathbb{P}^2 ; these are height one primes on C .



This is an easy result in homological algebra; what is difficult is showing that $H^1(C, \mathcal{L}(D_W)) = H^0(C, \mathcal{L}(K_C - D_W))$.

6 February 27, 2009 (Speaker: Brian Harbourne)

6.1 Proof of Riemann Roch.

We will produce a proof of the Riemann Roch theorem for curves (over \mathbb{C}).

Proof. Case 1. Let D be an effective divisor on a smooth projective curve C . Let $D = \sum_{i=1}^m P_i$ be a Weil divisor. Look at the exact sequence of sheaves and tensor

$$\left(0 \longrightarrow \mathcal{O}_C(-D) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_D \longrightarrow 0 \right) \otimes_{\mathcal{O}_C} \mathcal{O}_C(D)$$

to obtain

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_D \longrightarrow 0.$$

Taking cohomology, we have

$$0 \longrightarrow \mathbb{C} \longrightarrow H^0(C, \mathcal{O}_C(D)) \longrightarrow \mathbb{C}^m \longrightarrow H^1(C, \mathcal{O}_C) \longrightarrow H^1(C, \mathcal{O}_C(D)) \longrightarrow 0$$

(this result uses that $m = \deg D$ and the Grothendik vanishing theorem).

Assume $H^0(C, \mathcal{O}_C(D))$ is positive with $\dim H^0 = h^0$. Working over \mathbb{C} , we obtain

$$\begin{aligned} h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) &= 1 - h^1(C, \mathcal{O}_C) + m \\ &= \deg D + 1 - h^1(C, \mathcal{O}_C) \\ &= \deg D + 1 - g_C. \end{aligned}$$

Case 2. Suppose D isn't an effective divisor; i.e., $D = \sum_{i=1}^r P_i - \sum_{i=1}^s Q_j$ with $A := \sum_{i=1}^r P_i$ and $B := \sum_{i=1}^s Q_j$. We can tensor the short exact sequence

$$\left(0 \longrightarrow \mathcal{O}_C(-B) \longrightarrow \mathcal{O}_C \longrightarrow \mathcal{O}_B \longrightarrow 0 \right) \otimes_{\mathcal{O}_C} A$$

to obtain

$$0 \longrightarrow \mathcal{O}_C(D) \longrightarrow \mathcal{O}_C(A) \longrightarrow \mathcal{O}_B \longrightarrow 0.$$

As before, we take cohomology and obtain the signed sum

$$\begin{aligned} h^0(C, \mathcal{O}_C(A)) - h^1(C, \mathcal{O}_C(A)) &= h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) + h^0(C, \mathcal{O}_B) \\ &= h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) + \deg B \end{aligned}$$

while the previous case gives

$$h^0(C, \mathcal{O}_C(A)) - h^1(C, \mathcal{O}_C(A)) = \deg A + 1 - g_C$$

(since A is an effective divisor). Rearranging, we have

$$h^0(C, \mathcal{O}_C(D)) - h^1(C, \mathcal{O}_C(D)) = \deg A - \deg B + 1 - g_C = \deg D + 1 - g_C.$$

□

6.2 Comparison of Affine Space and Projective Space.

Let $R = k[x_0, \dots, x_n]$ be graded in the usual way, and let $M = (x_0, \dots, x_n)R$ be the ideal generated by the indeterminates. Recall that for affine space $X = \text{Spec}(R)$, there is a bijective correspondence between ideals in R and closed subschemes of X . In projective space $X = \text{Proj}(R) = \mathbb{P}^n = \{\text{homogeneous primes } I \subsetneq (x_0, \dots, x_n)\}$, there is a bijective correspondence between saturated homogeneous ideals $I \subseteq (x_0, \dots, x_n)$ and closed subschemes of P .

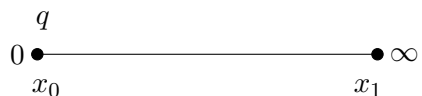
Definition. The **saturation** of a homogeneous ideal J is defined

$$\text{sat}(J) = \bigcup_{i \geq 1} (J : M^i) = \{f \in R \mid fx_i^i \in J \forall i = 0, \dots, n, i \geq 1\}.$$

If $J = \text{sat}(J)$, we say J is **saturated**.

Example. Let $\mathbb{P}^1 = \text{Proj}(\mathbb{C}[x_0, x_1])$.

Figure 5: The Projective Line



The point q is a single point of \mathbb{P}^1 (a closed subscheme). What ideal corresponds to q ? The ideal $(x_0) \subseteq \mathbb{C}[x_0, x_1]$.

Suppose

$$I = (x_0) = 0 \oplus \langle x_0 \rangle \oplus \langle x_0^2, x_0x_1 \rangle \oplus \langle x_0^3, x_0^2x_1, x_0x_1^2 \rangle \oplus \dots$$

and

$$J = \langle x_0^2, x_0x_1 \rangle \oplus \langle x_0^3, x_0^2x_1, x_0x_1^2 \rangle \oplus \dots$$

Then $J = I \cap M^2$, and $x_0 \in \text{sat}(J)$ since $x_0x_0 \in J$ and $x_0x_1 \in J$. So $I \subsetneq \text{sat}(J)$, and hence J is not saturated.

Similarly, the ideal

$$J = (x_0^2) = 0 \oplus 0 \oplus \langle x_0^2 \rangle \oplus \langle x_0^3, x_0^2x_1 \rangle \oplus \dots$$

is not saturated but defines q nonetheless.

7 March 6, 2009 (Speaker: Brian Harbourne)

Let $R = k[\mathbb{P}^n] = k[x_0, \dots, x_n] = \bigoplus_{t \geq 0} R_t$, where R_t is the vector space span of homogeneous forms of degree t .

Remark. There exists a bijective correspondence between each of the following:

- (1) Saturated homogeneous ideals $I \subsetneq R$
- (2) Closed subschemes $Z \subseteq \mathbb{P}^n$
- (3) Ideal sheaves $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n}$

The definition of a subscheme gives (2) \Leftrightarrow (3). (It remains to show (1) \Leftrightarrow (3).)

Suppose $I \subsetneq R$ is any homogeneous ideal, and $I \mapsto \tilde{I} \subseteq \mathcal{O}_{\mathbb{P}^n}$. Say $I = (F_1, \dots, F_r)$ for homogeneous forms F_i . Define $d_i = \deg F_i$ and $U_i = \text{Spec}(k[\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}])$. We have that U_i is an open subset of $\mathbb{P}^n = \text{Proj}(R)$ and $\bigcup_{i \geq 0} U_i = \mathbb{P}^n$. Now $\tilde{I}(U_j) = (f_{1j}, \dots, f_{rj}) \subseteq k[\frac{x_0}{x_j}, \dots, \frac{x_n}{x_j}]$, where $f_{ij} = \frac{F_i}{x_j^{d_i}}$. This is a homogeneous rational form of degree zero, or the polynomial obtained by $F_i(\frac{x_0}{x_j}, \dots, x_n x_j)$.

Given $\mathcal{I} \subseteq \mathcal{O}_{\mathbb{P}^n}$, pick any hyperplane $H \subseteq \mathbb{P}^n$. Let $\mathcal{O}_{\mathbb{P}^n}(t) = \mathcal{L}(t+1)$ for some $t \in \mathbb{Z}$, and let $\mathcal{I}(t) = \mathcal{I} \otimes_{\mathcal{O}_{\mathbb{P}^n}} \mathcal{O}_{\mathbb{P}^n}(t)$.

Given \mathcal{I} , we obtain I (which is saturated and homogeneous as follows):

$$\begin{array}{ccc} \bigoplus_{t \geq 0} \Gamma(\mathbb{P}^n, \mathcal{I}(t)) & \xrightarrow{\subseteq} & \Gamma(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(t)) \\ \downarrow \cong & & \cong \downarrow \\ \text{sat}(I) & \hookrightarrow & \bigoplus_{t \geq 0} R_t = R \end{array}$$

Example. Given \mathbb{P}^1 , $k[\mathbb{P}^1] = k[x, y]$, and the hyperplane H defined by $x = 0$ (i.e., H corresponds to the prime ideal $(x) \subseteq k[x, y]$), $\mathcal{O}_{\mathbb{P}^1}(1)$ denotes $\mathcal{L}(H)$. Let $D_W = H$, $\mathcal{I} = \mathcal{O}_{\mathbb{P}^1}(-D_W) \subseteq \mathcal{O}_{\mathbb{P}^1}$. Then $\mathcal{I}(t) = \mathcal{L}(tH - H) = \mathcal{L}((t-1)H) = \mathcal{O}_{\mathbb{P}^1}(t-1)$. So

$$\begin{aligned} \Gamma(\mathbb{P}^1, \mathcal{I}(H)) &= \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t-1)) \\ &= \Gamma(\mathbb{P}^1, \mathcal{L}((t-1)H)) \\ &= \frac{R_{t-1}}{x^{t-1}} \\ &\subseteq \Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(t)) = \Gamma(\mathbb{P}^1, \mathcal{L}(tH)) = \frac{R_t}{x^t}. \end{aligned}$$

Indeed, this follows because $\frac{R_{t-1}}{x^{t-1}} = \frac{xR_{t-1}}{x^t} \subseteq \frac{R_t}{x^t}$.

The commutative square we're interested in is as below, where the isomorphism is multiplication by x^t .

$$\begin{array}{ccc}
 \bigoplus_{t \geq 0} \frac{xR_{t-1}}{x^t} \subseteq & \hookrightarrow & \bigoplus_{t \geq 0} \frac{R_t}{x^t} \\
 \downarrow \cong & & \downarrow \cong \\
 \bigoplus_{t \geq 0} xR_{t-1} \subseteq & \hookrightarrow & \bigoplus_{t \geq 0} R_t
 \end{array}$$

A sheaf of ideals \mathcal{I} on a scheme X can be turned into a line bundle on the “blow-up” scheme \tilde{X} . Let \mathcal{I} be an ideal sheaf on $X = \mathbb{P}^n$. Let $Y \subseteq X$ be the subscheme of X defined by \mathcal{I} . The blow-up scheme \tilde{X} comes with a morphism $\tilde{X} \rightarrow X$.

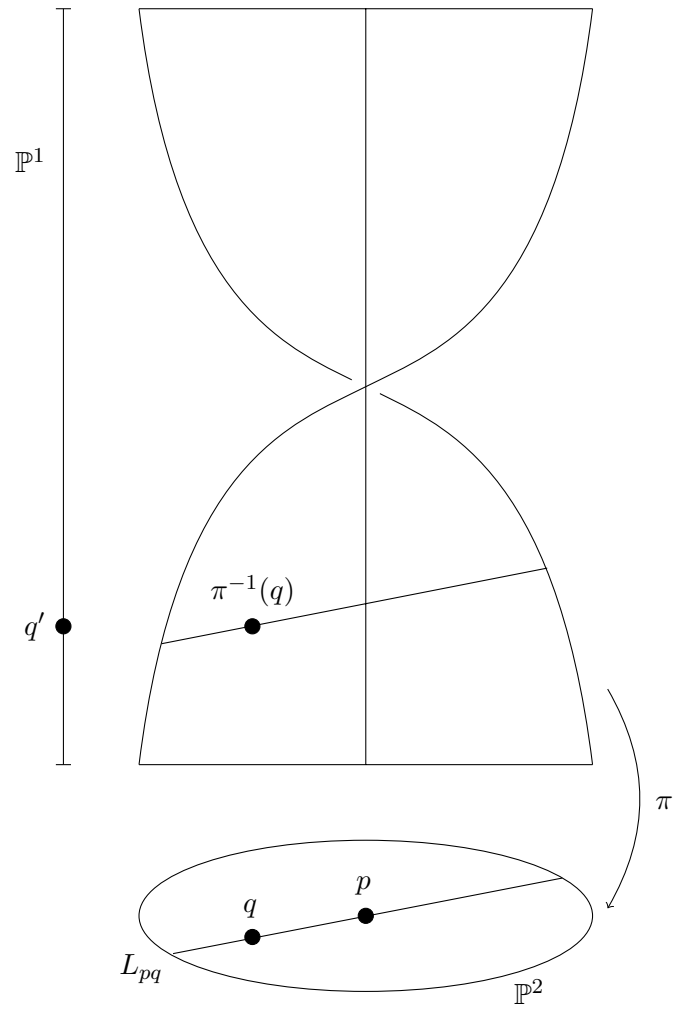
Theorem. Hartshorne II.7. Let $\delta = \bigoplus_{d \geq 0} \mathcal{I}^d$ be a graded sheaf of \mathcal{O}_X algebras. Then $\text{Proj}(S) = \tilde{X} \xrightarrow{\pi} X$.

Remark. The object $D = \pi^{-1}(Y) \subseteq \tilde{X}$ is a divisor, and $\mathcal{O}_{\tilde{X}}(D) = \tilde{\mathcal{I}} = \pi^{-1}(\mathcal{I}(\mathcal{O}_X))$.

Example. Let's blow up a point of $X = \mathbb{P}^2$ (not much happens in \mathbb{P}^1 ; a point is already a divisor). Suppose $p \in X$. We have the commutative square

$$\begin{array}{ccc}
 \tilde{X} & \xrightarrow{\pi} & X \\
 \downarrow \subseteq & & \downarrow \subseteq \\
 \mathbb{P}^1 \times \mathbb{P}^2 & \longrightarrow & \mathbb{P}^2.
 \end{array}$$

Figure 6: The fiber over p is the divisor D .



8 March 13, 2009 (Speaker: Brian Harbourne)

Let $P = (a, b, c) \in \mathbb{P}^2$, $R = k[\mathbb{P}^2] = k[x, y, z]$. Let $I(p)$ be the ideal generated by all forms F such that $F(p) = 0$; that is, $I(p) = (bx - ay, cx - az, cy - bz)$ is generated by the determinants of the 2×2 minors of the matrix $\begin{pmatrix} x & y & z \\ a & b & c \end{pmatrix}$.

Let p_1, \dots, p_r be distinct points of \mathbb{P}^2 , and let $m_1, \dots, m_r \geq 0$. Let $I = \bigcap_{i=1}^r (I(p_i)^{m_i})$. This is a saturated ideal, in fact. It defines a subscheme $Z \subseteq \mathbb{P}^2$ called a **fat point subscheme**. The adjective “fat” refers to the fact that the exponents can be large, thereby yielding extra structure.

Denote Z by $Z = m_1 p_1 + \dots + m_r p_r$. Let $\tilde{R} = \mathcal{O}_{\mathbb{P}^2}$ (sheaf). Let $\mathcal{I}_Z = \tilde{I}$ (sheaf of ideals in $\mathcal{O}_{\mathbb{P}^2}$ defining Z). Let $\mathcal{I}_Z(t) = \mathcal{I}_Z \otimes_{\mathcal{O}_{\mathbb{P}^2}} \mathcal{O}_{\mathbb{P}^2}(t)$ (pick a line; this is the line bundle corresponding to twisting the ideal sheaf by t).

Remark. There exists a commutative diagram

$$\begin{array}{ccc} I = \bigoplus_{t \geq 0} I_t & \xrightarrow{\cong} & \bigoplus_{t \geq 0} H^0(\mathbb{P}^2, \mathcal{I}_Z(t)) \\ \subseteq \downarrow & & \downarrow \subseteq \\ R = \bigoplus_{t \geq 0} R_t & \xrightarrow{\cong} & \bigoplus_{t \geq 0} H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(t)) \end{array}$$

with degree preserving isomorphisms.

Example. When $n = 2$, $\dim_k R_t = \binom{t+2}{2}$. For $R = k[\mathbb{P}^n]$, $\dim_k R_t = \binom{t+n}{n}$.

Example. Let $Z = 1 \cdot p$; $I = I(p)$. Let $p = (0, 0, 1)$. We can verify that $I(p) = (x, y)$ by checking the determinants of the 2×2 minors of the matrix $\begin{pmatrix} x & y & z \\ 0 & 0 & 1 \end{pmatrix}$. Now $I(p)_t^m$ is the k -vector space span of all monomials $x^i y^j z^l$ such that $i + j + l = t$ and $i + j \geq m$. Then $\dim I(p)_t^m = \binom{t+2}{2} - \binom{m+1}{2}$ is the number of such monomials.

Suppose instead $Z = m_1 p_1 + \dots + m_r p_r$. Here $\dim_k I(Z)_t \geq \binom{t+2}{2} - \left(\sum_{i=1}^r \binom{m_i+1}{2} \right)$ (compare to Riemann’s inequality).

Remark.

(1) In general, $\dim_k I(Z)_t = h^0(\mathbb{P}^2, \mathcal{I}_Z(t)) \geq \binom{t+2}{2} - \left(\sum_{i=1}^r \binom{m_i+1}{2} \right)$.

(2) If $t \geq 0$, $h^0(\mathbb{P}^2, \mathcal{I}_Z(t)) - h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) = \binom{t+2}{2} - \left(\sum_{i=1}^r \binom{m_i+1}{2} \right)$.

$$(3) \text{ For any integer value of } t, h^0(\mathbb{P}^2, \mathcal{I}_Z(t)) - h^1(\mathbb{P}^2, \mathcal{I}_Z(t)) + h^2(\mathbb{P}^2, \mathcal{I}_Z(t)) = \frac{t^2 - 3t + 2}{2} - \left(\sum_{i=1}^r \frac{m_i(m_i + 1)}{2} \right).$$

8.1 Making contact with Line Bundles.

We can blow up the points $p_1, \dots, p_r \in \mathbb{P}^2$. This gives a morphism $\pi : X \rightarrow \mathbb{P}^2$ where $|\pi^{-1}(p)| = 1$ unless $p \in \{p_1, \dots, p_r\}$, and $\pi^{-1}(p_i) = E_i \cong \mathbb{P}^1$ for $i = 1, \dots, r$. Let L' be a general line. Let $L = \pi^{-1}(L')$.

Remark. Up to equivalence, every divisor is of the form $tL - \sum_{i=1}^r m_i E_i$ for $m_i, t \in \mathbb{Z}$; i.e., L, E_1, \dots, E_r are divisors on D . Then for $t \geq 0, m_i \geq 0$, $H^0(X, \mathcal{O}_X(tL - \sum_{i=1}^r m_i E_i)) \cong H^0(\mathbb{P}^2, \mathcal{I}_Z(t))$ where $Z = m_1 p_1 + \dots + m_r p_r$.

8.2 Riemann-Roch for Surfaces.

Let D be a divisor on a surface X . Then Riemann-Roch for surfaces says

$$\begin{aligned} h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) + h^2(X, \mathcal{O}_X(D)) \\ = \frac{D \cdot D - K_X \cdot D}{2} + h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X). \end{aligned}$$

In this formula, $D \cdot D = \deg(\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_D) = \deg \mathcal{O}_D$ and K_X is the canonical class on the surface.

Proof sketch. For a smooth curve D , consider the short exact sequence

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{O}_X(D) \longrightarrow \mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_D \longrightarrow 0$$

and take cohomology. The signed sum for $\mathcal{O}_X(D)$ is $h^0(X, \mathcal{O}_X(D)) - h^1(X, \mathcal{O}_X(D)) + h^2(X, \mathcal{O}_X(D))$, the signed sum for \mathcal{O}_X is $h^0(X, \mathcal{O}_X) - h^1(X, \mathcal{O}_X) + h^2(X, \mathcal{O}_X)$, and the signed sum for $\mathcal{O}_X(D) \otimes_{\mathcal{O}_X} \mathcal{O}_D$ is $\frac{D \cdot D - K_X \cdot D}{2}$. □

9 March 27, 2009 (Speaker: Susan Cooper)

9.1 Introduction to Hilbert functions and first differences for points.

Let $R = k[x_0, \dots, x_n] = \bigoplus_{i \geq 0} R_i = k[\mathbb{P}^n]$, where $k = \bar{k}$ and $\text{char } k = 0$. Recall that R_i is the vector space generated by monomials of degree i . Let $X = \{p_1, \dots, p_s\}$ be a collection of s distinct points. Let $I = \mathcal{I}(X) = P_1 \cap \dots \cap P_s$ be a homogeneous ideal, and let P_i denote the prime ideal of height n corresponding to $\mathcal{I}(p_i)$. Then $\text{ht}(I) = n$. Fix $A = R/I$, the homogeneous coordinate ring of X .

Remark. The ring A is graded;

$$A = \bigoplus_{d \geq 0} A_d = \bigoplus_{d \geq 0} (R/I)_d = \bigoplus_{d \geq 0} R_d/I_d.$$

Definition. The **Hilbert function of X** is the function $H(X, \bullet) : \mathbb{N}_{\geq 0} \rightarrow \mathbb{N}_{\geq 0}$ where

$$H(x, t) = \dim_k(A_t) = \dim_k R_t - \dim_k I_t = \binom{n+t}{t} - \dim_k I_t.$$

The **first difference of $H(X)$** , denoted $\Delta H(X)$, is the sequence $\{\Delta H(X, t)\}_{t \geq 0}$ where $\Delta H(X, t) = H(X, t) - H(X, t-1)$ for $t \geq 1$ and $\Delta H(X, 0) = 0$.

Remark. Most linear forms miss any finite fixed set of points, which implies that there is a linear form F not contained in $\bigcup_{i=1}^s P_i$. Thus a general linear form in R is not a zerodivisor of A . We can make a linear change of variables so that x_0 does not become a zerodivisor on $A = R/I$. Let $B = A/(\bar{x}_0)$.

Remarks.

- (1) The ring $B \cong k[x_1, \dots, x_n]/J$, where J is the homogeneous ideal of $k[x_1, \dots, x_n]$ obtained by setting $x_0 = 0$ in all elements of I , since $B \cong R/(x_0, I) = k[x_0, \dots, x_n]/(x_0, I)$.
- (2) The ring B is graded.
- (3) The Krull dimension of B is 0.
- (4) For the ideal J as in (1), $\sqrt{J} = (x_1, \dots, x_n)$.
- (5) $B_t = 0$ for $t \gg 0$.

(6) $B_r = 0$ implies $B_{r+1} = 0$.

We refer to items (5) and (6) as (\star) .

Remark. The map $(\overline{x_0}) : A_d \rightarrow A_{d+1}$ via $a \mapsto \overline{x_0}a$ is injective since x_0 is a non-zerodivisor. Therefore $\dim_k A_{d+1} \geq \dim_k A_d$ for all d $(\star\star)$. In fact, $\overline{x_0}A_d = (\overline{x_0})_{d+1}$. So as a vector space,

$$A_{d+1}/(\overline{x_0})A_d = A_{d+1}/(\overline{x_0})_{d+1} \cong B_{d+1}.$$

Corollary. It follows from (\star) that for $t \gg 0$, $(\overline{x_0}) : A_d \rightarrow A_{d+1}$ is an isomorphism, and if $(\overline{x_0})A_r \rightarrow A_{r+1}$ is an isomorphism, then $(\overline{x_0}) : A_{r+1} \rightarrow A_{r+2}$ is also an isomorphism.

Proposition. The quantity $\Delta H(X, t) = \dim_k B_t$.

Proof. We have that $H(X, t) = \dim_k A_t$ and $H(X, t-1) = \dim_k A_{t-1}$. Since $(\overline{x_0}) : A_{t-1} \rightarrow A_{t-2}$ is injective, we have $\dim_k A_{t-1} = \dim_k ((\overline{x_0})A_{t-1})$. Now $B_t \cong A_t / ((\overline{x_0})A_{t-1})$, and therefore

$$\begin{aligned} \dim_k B_t &= \dim_k A_t \\ &= \dim_k ((\overline{x_0})A_{t-1}) \\ &= \dim_k A_t - \dim_k A_{t-1} = \Delta H(X, t). \end{aligned}$$

□

Corollary. There exists an integer $d > 0$ such that

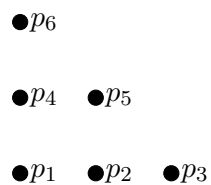
- (1) $H(X, t) = d$ for all $t \gg 0$,
- (2) $H(X, t) = d$ implies $H(X, t+1) = d$, and
- (3) $H(X, t) = d$ and $H(X, t+1) = c$ implies $c = d$.

Example. Consider the set of points $X \subseteq \mathbb{P}^2$ as represented in Figure 7; here $R = k[x_0, x_1, x_2]$ and $I = \mathcal{I}(X)$.

It follows from Bezout's Theorem (Hartshorne Cor. 78) that $I_0 = I_1 = I_2 = 0$. Now $H(X, 0) = 1$, $H(X, 1) = 3$, and $H(X, 2) = 6$. To calculate $H(X, t)$ for $t > 2$, we must determine $\dim_k I_t$ for $t > 2$. For $t = 3$, suppose $F \in I_3$. Then F has $\binom{2+3}{3} = 10$ coefficients. Now $F(p_i) = 0$ imposes a linear condition on the coefficients, and so we have homogeneous system of six equations with 10 unknowns. The number of independent solutions is exactly $10 - \text{rank } A$ where A is the coefficient matrix for the system, and

Figure 7:

X :



rank $A \leq 6$. Therefore the number of independent solutions is at most 4. Therefore $H(X, 3) = \binom{2+3}{3} - \dim_k I_3 \leq 10 - 4 = 6$. But $(\star\star)$ implies $H(X)$ is nondecreasing, and $H(X, 2) = 6$; therefore $H(X, 3) = 6$. By the last corollary, $H(X, t) = 6$ for all $t \geq 2$. Therefore $H(X) = (1, 3, 6, 6, \dots)$ and $\Delta H(X) = (1, 2, 3, 0, \dots)$.

10 April 10, 2009 (Speaker: Susan Cooper)

Let $R = k[x_0, \dots, x_n]$, $k = \bar{k}$. Let $X = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_k^n$ be a set of distinct points, and let $I = \mathcal{I}(X)$.

Lemma. With X as above,

(1) $H(X, d) \leq s$ for all d , and

(2) $H(X, s-1) = s$.

Proof. For (1), if $F \in R_d$, then F has $\binom{n+d}{d}$ coefficients. Then $F \in I$ implies $F(p_i) = 0$ for $1 \leq i \leq s$. Now $F(p_i) = 0$ imposes a linear condition on the coefficients of F . This gives a homogeneous linear system with s equations in $\binom{n+d}{d}$ unknowns. The number of linearly independent solutions is $\binom{n+d}{d} - \text{rank } A$, where A is the $s \times \binom{n+d}{d}$ coefficient matrix. It follows that $\dim_k I_d \geq \binom{n+d}{d} - \min \left\{ s, \binom{n+d}{d} \right\}$. Thus

$$H(X, d) = \binom{n+d}{d} - \dim_k I_d \leq \min \left\{ s, \binom{n+d}{d} \right\} \leq s.$$

For (2), suppose H_i is a hyperplane in \mathbb{P}_k^n such that $H_i(p_i) = 0$ and $H_i(p_j) \neq 0$ for any $j \neq i$. Notice that since $|k| = \infty$, such an H_i exists. Let $F_i = \prod_{j \neq i}^s H_j$ for $1 \leq i \leq s$. Then $F_1, \dots, F_s \in R_{s-1}$ and $F_i \notin I$ since $F_i(p_i) \neq 0$ for all i .

We claim that $\overline{F_1}, \dots, \overline{F_s}$ are linearly independent of A_{s-1} where $A = R/I$. Suppose c_1, \dots, c_s are scalars such that $\sum_{i=1}^s c_i F_i = 0$. Then $c_1 F_1 + \dots + c_s F_s \in I$. Suppose $c_l \neq 0$. Since $c_1 F_1 + \dots + c_s F_s \in I$, $(c_1 F_1 + \dots + c_s F_s)(P_l) = 0$. But $F_j(P_l) = 0$ for all $j \neq l$ and $F_l(P_l) \neq 0$, a contradiction. Therefore $c_1 = \dots = c_s = 0$.

Therefore $H(X, s-1) = \dim_k A_{s-1} \geq s$, but by (1), $H(X, s-1) \leq s$. Therefore $H(X, s-1) = s$. \square

Corollary. With X as above,

(1) $H(X, t) = s$ for all $t \gg 0$.

(2) $H(X, t) = s$ implies $H(X, t+1) = s$.

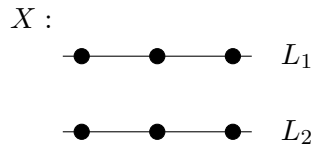
(3) $H(X, t) = c$ and $H(X, t+1) = c$ implies $c = |X| = s$.

Remark. Note that $\sum_{t=0}^{d-1} \Delta H(X, t) = H(X, d)$ implies $\sum_{t=0}^{\infty} \Delta H(X, t) = s = |X|$.

Example. Suppose X contains 6 points in \mathbb{P}^2 , and let $I = \mathcal{I}(X)$ and $R = k[x_0, x_1, x_2]$.

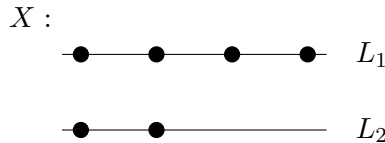
- (a) Last time, we considered the scenario in Figure 7.
- (b) Suppose X is arranged as in Figure 8. Then $I_0 = I_1 = 0$, so $H(X, 0) = \binom{2-0}{0} - 0 = 1$ and $H(X, 1) = \binom{2+1}{1} - 0 = 3$. By Bezout's theorem, there is no conic passing through X , and therefore $\dim_k I_2 = 1$, so $H(X, 2) = \binom{2+2}{2} - 1 = 5$. Since $H(X)$ is nondecreasing, $H(X, 3)$ must be 5 or 6. But by the corollary, $|X| = 5$ is a contradiction. Therefore $H(X, 3) = 6$, and hence $H(X, t) = 6$ for $t \geq 3$. Therefore $H(X) = (1, 3, 5, 6, 6, \dots)$, and $\Delta H(X) = (1, 2, 2, 1, 0, 0, \dots)$.

Figure 8:



- (c) (Exercise). Let X be arranged as in Figure 9. Determine $H(X)$ and $\Delta H(X)$.

Figure 9:

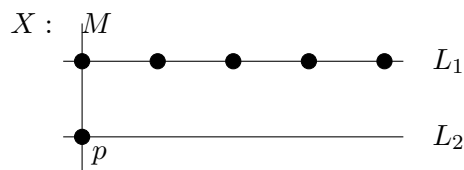


Solution. As before, $H(X) = (1, 3, 5, 6, 6, \dots)$ and $\Delta H(X) = (1, 2, 2, 1, 0, 0, \dots)$.

- (d) Let X be arranged as in Figure 10. Again, $I_0 = I_1 = 0$, so $H(X, 0) = 1$ and $H(X, 1) = 3$. But I_2 has a k -basis $\{L_1L_2, L_1M\}$, so $\dim_k I_2 = 2$. Therefore $H(X, 2) = 4$. But $H(X, 3) \neq 4$ else $|X| = 4$, a contradiction.

If $H \in I_3$, Bezout's theorem implies $H = L_1F$ where F has degree 2 and $F(p) = 0$. We know that F has $\binom{2+2}{2} = 3$ coefficients, so we get a homogeneous linear system with 1 equation and 3 unknowns. There are at least 5 such linearly independent forms. Thus $\dim_k I_3 \geq 5$. Then $H(X, 3) \leq \binom{2+3}{3} - 5 = 5$, and hence $H(X, 3) = 5$. By the corollary, $H(X, t) = 6$ for all $t \geq 6$. So $H(X) = (1, 3, 4, 5, 6, 6, \dots)$ and $\Delta H(X) = (1, 2, 1, 1, 1, 0, 0, \dots)$.

Figure 10:



- (e) (Exercise) Verify that if X is a set of six colinear points, the $H(X) = (1, 2, 3, 4, 5, 6, 6, \dots)$.

Question. If $X = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_k^n$ where the p_i are distinct, what can $H(X)$ be? For example, does there exist a set X of six points in \mathbb{P}^2 with Hilbert function $(1, 2, 3, 5, 6, 6, \dots)$ or $(1, 2, 4, 6, 6, \dots)$?

11 April 17, 2009 (Susan Cooper)

Question. Which sequences of numbers arise as the Hilber function of some finite set of distinct points in \mathbb{P}_k^n ?

11.1 Some enumeration

Proposition. Let h, i be positive integers. We can uniquely write $h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j}$ where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$.

Definition. We call the decomposition above the *i -binomial expansion of h* . We define $h^{(i)} := \binom{m_i+1}{i+1} + \binom{m_{i-1}+1}{i} + \cdots + \binom{m_j+1}{j+1}$ and $0^{(i)} = 0$ by convention.

Example. The 3-binomial expansion of 18 is

$$18 = \binom{5}{3} + \binom{4}{2} + \binom{2}{1}$$

and

$$18^{(i)} = \binom{6}{4} + \binom{5}{3} + \binom{3}{2} = 28.$$

Definition. Let $d = \{d_i\}_{i \geq 0}$ be a sequence of nonnegative integers. We say d is an *O -sequence* if $d_0 = 1$ and $d_{i+1} \leq d_i^{(i)}$ for all $i \geq 1$. We say d is a **differentiable O -sequence** if both d and Δd are O -sequences. Recall $\Delta d = \{c_i\}_{i \geq 0}$ where $c_0 = d_0$ and $c_i = d_i - d_{i-1}$ for $i \geq 1$.

Example.

- (1) The sequence $d = (1, 2, 3, 5, 6, 6, \dots) = \{d_i\}_{i \geq 0}$ is not an O -sequence since $5 \not\leq 3^{(2)} = 4$.
- (2) The sequence $d = (1, 3, 6, 8, 9, 11, 11, \dots) = \{d_i\}_{i \geq 0}$ is an O -sequence, but not a differentiable one: $\Delta d = (1, 2, 3, 1, 2, 0, 0, \dots) = \{c_i\}_{i \geq 0}$ has $2 \not\leq 1^{(4)} = 1$.
- (3) The sequence $d = (1, 3, 4, 5, 6, 6, \dots)$ is a differentiable O -sequence since $\Delta d = (1, 2, 1, 1, 1, 0, \dots)$ is an O -sequence.

Theorem. (Geramita-Maroscia-Roberts) Let $\mathcal{H} = \{h_t\}_{t \geq 0}$ be a sequence of nonnegative integers. Then the following are equivalent:

- (1) $\mathcal{H} = H(x)$ for some finite set $X \subseteq \mathbb{P}_k^n$ of s distinct points p_1, \dots, p_s .
- (2) \mathcal{H} is a differentiable O -sequence where $h_1 \leq n + 1$ and $h_t = s$ for $t \gg 0$.

Example. The sequence $\mathcal{H} = (1, 3, 6, 8, 9, 11, 11, \dots)$ is not a differentiable O -sequence, so there is no set of 11 distinct points in \mathbb{P}_k^n for $n \geq 2$ with such a Hilbert function (in fact, there is no such set of 11 points in \mathbb{P}_k^1 , either). The sequence $\mathcal{H} = (1, 4, 10, 19, 30, 40, 50, 50, \dots)$ is a differentiable O -sequence, and hence there exist 50 distinct points in \mathbb{P}_k^3 with such a Hilbert function.

11.2 What good is the O -sequence approach?

Definition. Let $d = \{d_i\}_{i \geq 0}$ be an O -sequence. We say d has **maximal growth in degree t** if $d_{t+1} = d_t^{(t)}$.

Remark. Let $X = \{p_1, \dots, p_s\} \subseteq \mathbb{P}_k^n$ be a set of s distinct points. Then $\Delta H(X)$ is an O -sequence. If $\Delta H(X)$ has maximal growth in some degrees, then we can sometimes obtain nontrivial geometric information about the subsets of X .

Example. Let $X \subseteq \mathbb{P}_k^2$ be any set of 21 distinct points with $H(X) = (1, 3, 6, 10, 14, 16, 18, 20, 21, \dots) = \{d_i\}_{i \geq 0}$. Note that $\Delta H(X) = (1, 2, 3, 4, 4, 2, 2, 2, 1, 0, 0, \dots) = \{c_i\}_{i \geq 0}$ has maximal growth in degree 5 since $c_6 = 2 = \binom{5}{5} + \binom{4}{4} = c_5^{(5)}$.

(E. D. Davis, 1985) Any set X as above contains a subset of 16 points lying on a conic in \mathbb{P}_k^2 .

Example. Suppose $X \subseteq \mathbb{P}_k^3$ is any set of 104 distinct points with $H(X) = (1, 4, 10, 20, 34, 48, 64, 79, 96, 100, 104, 104, \dots)$ such that no four of the points lie on a plane, and note $\Delta H(X) = (1, 3, 6, 10, 14, 14, 16, 15, 17, 4, 4, 0, 0, \dots) = \{c_i\}_{i \geq 0}$ has maximal growth in degree 9 since $4 = 4^{(9)}$.

(Bigatti-Geramita-Migliore, 1994) Exactly 89 points of a set X as above lie on a quadric surface where 40 or 41 of these points lie on an irreducible curve of degree 4. The remaining 15 points have exactly 13 lying on a rational normal curve in \mathbb{P}_k^3 .

12 April 24, 2009 (Brian Harbourne)

Let $p \in \mathbb{P}^2$. Let $I(p) \subseteq k[\mathbb{P}^2]$ be the ideal generated by all forms that vanish at p .

Definition. Let $F \in k[\mathbb{P}^2]$ be homogeneous and nonzero. Then the multiplicity of F at p , $\text{mult}_p F$, is m if and only if $F \in I(p)^m \setminus I(p)^{m+1}$, where $I(p)^0 = k[\mathbb{P}^2]$.

Remark. $F \in I(p)^m$ if and only if $\text{mult}_p F \geq m$.

Example. For $k[\mathbb{P}^2] = k[x, y, z]$, say $p = (0, 0, 1)$; i.e., $I(p) = (x, y)$. Say $F = \sum_{i+j+\ell=d} c_{ij\ell} x^i y^j z^\ell$ is nonzero, $c_{ij\ell} \in k$. Then $\text{mult}_p F = \min\{i + j \mid c_{ij\ell} \neq 0\}$. For example, if $F = x^2 + xy + yz$, then $\text{mult}_p F = 1$ since c_{011} has $i + j = 1$ and $c_{011} = 1$. Alternatively, set $z = 1$ and find the term of $F(x, y, 1)$ of least degree.

Remark. Say F, G are forms such that $FG \neq 0$. then $\text{mult}_p FG = \text{mult}_p F + \text{mult}_p G$.

12.1 Open Problem #1

Let $m_1, \dots, m_r \geq 0$ and let p_1, \dots, p_r be distinct points, $k = \bar{k}$. Let $z = m_1 p_1 + \dots + m_r p_r$ be in the free abelian group on points of \mathbb{P}_k^2 . Let $I(z) = \bigcap_{i=1}^r I(p_i)^{m_i}$ be the fat point ideal of z . This is a homogeneous ideal. Let α_z be the least degree t such that $I(z)_t \neq 0$. Alternatively, α_z is the least t such that $(x, y, z)^t \supseteq I(z)$ but $(x, y, z)^{t+1} \not\supseteq I(z)$. So α_z is the (x, y, z) -adic order of $I(z)$. Let $\alpha(m_1, \dots, m_r) = \max \alpha_z$ where the maximum is taken over all sets of distinct points p_1, \dots, p_r of \mathbb{P}^2 .

Example. Suppose $z = p_1 + p_2 + p_3$. Then $\alpha(1, 1, 1) = 2$ because no matter how the points are arranged, there is always a conic through the points, but not always a line.

Example. Suppose $z = 10p_1 + 3p_2 + 5p_3$. Then $\alpha_z \leq 10$ but $\alpha_z \geq \max m_i = 10$, and hence $\alpha_z = 10$.

Remark. For $m_i, m'_i \geq 0$, $\alpha(m_1, \dots, m_r) + \alpha(m'_1, \dots, m'_r) \geq \alpha(m_1 + m'_1, \dots, m_r + m'_r)$.

Open Question. Prove the following conjecture (Nagata, 1959):

$$\alpha(m_1, \dots, m_r) \geq \frac{\sum m_i}{\sqrt{r}}$$

if $r = 1$, $r = 4$, or $r \geq 9$.

Remark. In particular, for $r \geq 9$, the conjecture is that the inequality is strict. This question is known to be true if r is a square.

Example. (Counterexamples)

(i) For $r = 2$, $\alpha(1, 1) = 1 < 2/\sqrt{2}$.

(ii) For $r = 3$, $\alpha(1, 1, 0) = 1 < 2/\sqrt{3}$.

(iii) For $r = 5$, $\alpha(1, 1, 1, 1, 1) = 2 < 5/\sqrt{5}$.

(iv) For $r = 6$, $\alpha(1, 1, 1, 1, 1, 0) = 2 < 5/\sqrt{6}$.

(v) For $r = 7$, $\alpha(2, 1, 1, 1, 1, 1, 1) = 3 < 8/\sqrt{7}$.

(vi) For $r = 8$, $\alpha(3, 2, 2, 2, 2, 2, 2, 2) = 6 < 17/\sqrt{8}$.

Theorem. (Nagata, 1959) For $m_i \geq 0$, $\sum m_i > 0$, if $r = s^2$, $s \geq 4$, then

$$\alpha(m_1, \dots, m_r) > \frac{\sum m_i}{s}.$$

Remark. This theorem can be applied to find a negative solution to Hilbert's 14-th problem.

Remark. Nagata's conjecture is equivalent to the conjecture that, for $r > 10$, $\alpha(m, \dots, m) \geq \sqrt{r}m$.

Proof. Nagata's conjecture implies the second conjecture trivially. For the other implication, suppose Nagata's conjecture is false. Then $\alpha(m_1, \dots, m_r) < \frac{\sum m_i}{\sqrt{r}}$ for some set of m_i . Then $\alpha(m_2, \dots, m_r, m_1) < \frac{\sum m_i}{\sqrt{r}}$. Thus

$$\alpha(\sum m_i, \dots, \sum m_i) \leq \alpha(m_1, \dots, m_r) + \alpha(m_2, \dots, m_r, m_1) + \dots + \alpha(m_r, \dots, m_{r-1}) < \sqrt{r} \sum m_i,$$

so that the second conjecture is false. □

Remark. Another conjecture equivalent to the previous conjectures is that, for $r \geq 10$, $z = \sum m_i p_i$ where $\sum m_i > 0$, $m_i \geq 0$ for all i , given a set of general points $p_1, \dots, p_r \in \mathbb{P}_2$, then

$$\alpha(z) \geq \frac{\sum m_i}{\sqrt{r}}.$$

13 May 1, 2009 (Brian Harbourne)

Recall. Nagata's conjecture: If $p_1, \dots, p_n \in \mathbb{P}^2$ are sufficiently general points, and if $n \geq 0$, then there is no nonzero homogeneous polynomial F of degree at most $m\sqrt{n}$ such that $\text{mult}_{p_i} F \geq m$ for all i . More generally, the set of all degree t homogeneous polynomials $F \in k[x, y, z] = k[\mathbb{P}^2]$ vanishing with $\text{mult}_{p_i} F \geq m_i$ for distinct points p_1, \dots, p_r is a k -vector space, denoted $V_t(m_1p_1 + \dots + m_r p_r)$.

Open Question. Compute $\dim_k V_t(m_1p_1 + \dots + m_r p_r)$. This dimension attains a minimum on some open set $(p_1, \dots, p_r) \in (\mathbb{P}^2)^r$. The function $(p_1, \dots, p_r) \mapsto \dim_k V_t(\sum m_i p_i)$ is semicontinuous. But what is this minimum?

Conjecture. (Segre-Harbourne-Gimigliano-Hirschowitz) Let $m \geq 1$, $n \geq 10$, let p_1, \dots, p_n be general, and let $k = \bar{k}$. Then $\dim_k V_t(mp_1 + \dots + mp_n) = \max\{0, \binom{t+2}{2} - n\binom{m+1}{2}\}$.

Remark. In general,

$$\dim_k V_t(m_1p_1 + \dots + m_n p_n) \geq \binom{t+2}{2} - \sum \binom{m+i}{2}.$$

Equality need not hold.

Remark. The SHGH conjecture implies the Nagata conjecture. To see this, it is enough to show that if $t \geq m\sqrt{n}$, then $\binom{t+2}{2} - n\binom{m+1}{2} \leq 0$.