Homework Solutions – Week of September 30

Note: The exercises from Section 4.2 should be completed the week of October 7.

Section 4.1:

(5) We have

\[
A v = \begin{bmatrix}
3 & 0 & 0 \\
0 & 1 & -2 \\
1 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
1
\end{bmatrix} = \begin{bmatrix}
6 \\
-3 \\
3
\end{bmatrix} = 3v.
\]

So, by definition, \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 3\).

(6) We have

\[
A v = \begin{bmatrix}
0 & 1 & -1 \\
1 & 1 & 1 \\
1 & 2 & 0
\end{bmatrix}
\begin{bmatrix}
2 \\
-1 \\
-1
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0
\end{bmatrix} = 0v.
\]

So, by definition, \(v\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 0\).

(9) We want to find \(x \neq 0\) such that \(A x = 1x\). Equivalently, we want to find \(x \neq 0\) such that

\[
A x - 1x = \left( A - I \right) x = 0.
\]

So, we need to find the null space of \(A - I\). We form the associated augmented matrix and row reduce:

\[
\begin{bmatrix}
A - I & | & 0
\end{bmatrix} = \begin{bmatrix}
-1 & 4 & | & 0 \\
-1 & 4 & | & 0
\end{bmatrix} \rightarrow \begin{bmatrix}
1 & -4 & | & 0 \\
0 & 0 & | & 0
\end{bmatrix}.
\]

Solving the system, we have

\[
null(A - I) = \left\{ t \begin{bmatrix} 4 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.
\]

Any non-zero multiple of \(u = \begin{bmatrix} 4 \\ 1 \end{bmatrix}\) is an eigenvector of \(A\) with eigenvalue \(\lambda = 1\). In particular,

\[
Au = 1u
\]

which shows that \(u\) is an eigenvector of \(A\) with eigenvalue 1.
(10) We want to find $x \neq 0$ such that $Ax = 4x$. Equivalently, we want to find $x \neq 0$ such that

$$Ax - 4x = (A - 4I)x = 0.$$ 

So, we need to find the null space of $A - 4I$. We form the associated augmented matrix and row reduce:

$$[A - 4I | 0] = \begin{bmatrix} -4 & 4 & 0 \\ -1 & 1 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$ 

Solving the system, we have

$$null(A - 4I) = \left\{ t \begin{bmatrix} 1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$ 

Any non-zero multiple of $u = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ with eigenvalue $\lambda = 4$. In particular,

$$Au = 4u$$

which shows that $u$ is an eigenvector of $A$ with eigenvalue 4.

(11) We want to find $x \neq 0$ such that $Ax = -1x$. Equivalently, we want to find $x \neq 0$ such that

$$Ax - (-1)x = (A + I)x = 0.$$ 

So, we need to find the null space of $A + I$. We form the associated augmented matrix and row reduce:

$$[A + I | 0] = \begin{bmatrix} 2 & 0 & 2 & 0 \\ -1 & 2 & 1 & 0 \\ 2 & 0 & 2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$ 

Solving the system, we have

$$null(A + I) = \left\{ t \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} : t \in \mathbb{R} \right\}.$$
Any non-zero multiple of \( \mathbf{u} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} \) is an eigenvector of \( A \) with eigenvalue \( \lambda = -1 \). In particular,

\[
A \mathbf{u} = -\mathbf{1} \mathbf{u}
\]

which shows that \( \mathbf{u} \) is an eigenvector of \( A \) with eigenvalue -1.

(13) \( A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \) is the matrix of a reflection \( F \) in the \( y \)-axis. The only vectors that \( F \) maps parallel to themselves are vectors of the form \( \begin{bmatrix} 0 \\ s \end{bmatrix} \) and \( \begin{bmatrix} t \\ 0 \end{bmatrix} \).

Vectors of the form \( \mathbf{u} = \begin{bmatrix} 0 \\ s \end{bmatrix} \) are transformed to \( F(\mathbf{u}) = \begin{bmatrix} 0 \\ s \end{bmatrix} = 1 \mathbf{u} \) (i.e. these are eigenvectors with eigenvalue 1). Vectors of the form \( \mathbf{v} = \begin{bmatrix} t \\ 0 \end{bmatrix} \) are transformed to \( F(\mathbf{v}) = \begin{bmatrix} -t \\ 0 \end{bmatrix} = -1 \mathbf{v} \) (i.e. these are eigenvectors with eigenvalue -1). Therefore, we have the eigenspaces

\[
E_1 = \text{span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)
\]

and

\[
E_{-1} = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right).
\]

(17) \( A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \) is the matrix of the transformation \( T \) which stretches by a factor of 2 horizontally and a factor of 3 vertically. The only vectors that \( T \) maps parallel to themselves are vectors of the form \( \begin{bmatrix} s \\ 0 \end{bmatrix} \) and \( \begin{bmatrix} 0 \\ t \end{bmatrix} \). Vectors of the form \( \mathbf{u} = \begin{bmatrix} s \\ 0 \end{bmatrix} \) are transformed to \( T(\mathbf{u}) = \begin{bmatrix} 2s \\ 0 \end{bmatrix} = 2 \mathbf{u} \) (i.e. these are eigenvectors with eigenvalue 2). Vectors of the form \( \mathbf{v} = \begin{bmatrix} 0 \\ t \end{bmatrix} \) are transformed to
\[ T(v) = \begin{bmatrix} 0 & 3t \\ 3t & 0 \end{bmatrix} = 3v \] (i.e. these are eigenvectors with eigenvalue 3). Therefore, we have the eigenspaces

\[ E_2 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) \]

and

\[ E_3 = \text{span} \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right). \]

(25) We first find the eigenvalues for \( A \):

\[
\det(A - \lambda I) = \det \begin{bmatrix} 2 - \lambda & 5 \\ 0 & 2 - \lambda \end{bmatrix} = (2 - \lambda)^2.
\]

The eigenvalues are the roots of \((2 - \lambda)^2 = 0\). So, the only eigenvalue of \( A \) is \( \lambda = 2 \).

To find the eigenspace \( E_2 \), we need to find the null space of \( A - 2I \). We row reduce the associated augmented matrix:

\[
[A - 2I | 0] = \begin{bmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}.
\]

The solution set is \( E_2 \). We see that

\[ E_2 = \text{span} \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right), \]

and so \( \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \) is a basis for the eigenspace \( E_2 \).

(26) We first find the eigenvalues for \( A \):

\[
\det(A - \lambda I) = \det \begin{bmatrix} (1 - \lambda) & 2 \\ -2 & (3 - \lambda) \end{bmatrix} = (1 - \lambda)(3 - \lambda) + 4 = \lambda^2 - 4\lambda + 7.
\]

The eigenvalues are the roots of \( \lambda^2 - 4\lambda + 7 = 0 \). Since there are no real roots of this polynomial, \( A \) has no eigenvalues.
(27) We first find the eigenvalues for $A$:

\[
\det(A - \lambda I) = \det \begin{bmatrix}
(1 - \lambda) & 1 \\
-1 & (1 - \lambda)
\end{bmatrix} = (1 - \lambda)(1 - \lambda) + 1 = \lambda^2 - 2\lambda + 2.
\]

The eigenvalues are the roots of $\lambda^2 - 2\lambda + 2 = 0$. We see that $A$ has eigenvalues $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$.

To find the eigenspace $E_{1+i}$, we need to find the null space of $A - (1 + i)I$. We row reduce the associated augmented matrix:

\[
[A - (1 + i)I \mid 0] = \begin{bmatrix}
-i & 1 & 0 \\
-1 & -i & 0
\end{bmatrix} \to \begin{bmatrix}
1 & i & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The solution set is $E_{1+i}$. We see that

\[E_{1+i} = \text{span} \left( \begin{bmatrix} -i \\ 1 \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} 1 \\ i \end{bmatrix} \right),\]

and so \(\left\{ \begin{bmatrix} 1 \\ i \end{bmatrix} \right\} \) is a basis for the eigenspace $E_{1+i}$.

To find the eigenspace $E_{1-i}$, we need to find the null space of $A - (1 - i)I$. We row reduce the associated augmented matrix:

\[
[A - (1 - i)I \mid 0] = \begin{bmatrix}
i & 1 & 0 \\
-1 & i & 0
\end{bmatrix} \to \begin{bmatrix}
1 & -i & 0 \\
0 & 0 & 0
\end{bmatrix}.
\]

The solution set is $E_{1-i}$. We see that

\[E_{1-i} = \text{span} \left( \begin{bmatrix} i \\ 1 \end{bmatrix} \right) = \text{span} \left( \begin{bmatrix} 1 \\ -i \end{bmatrix} \right),\]

and so \(\left\{ \begin{bmatrix} 1 \\ -i \end{bmatrix} \right\} \) is a basis for the eigenspace $E_{1-i}$.

(28) We first find the eigenvalues for $A$:

\[
\det(A - \lambda I) = \det \begin{bmatrix}
(2 - \lambda) & -3 \\
1 & (-\lambda)
\end{bmatrix} = (2 - \lambda)(-\lambda) + 3 = \lambda^2 - 2\lambda + 3.
\]
The eigenvalues are the roots of \( \lambda^2 - 2\lambda + 3 = 0 \). We see that \( A \) has eigenvalues \( \lambda_1 = 1 + \sqrt{2}i \) and \( \lambda_2 = 1 - \sqrt{2}i \).

To find the eigenspace \( E_{1+\sqrt{2}i} \), we need to find the null space of \( A - (1 + \sqrt{2}i)I \). We row reduce the associated augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 3 \\
5 & 1 & 1 \\
0 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The solution set is \( E_{1+\sqrt{2}i} \). We see that

\[
E_{1+\sqrt{2}i} = \text{span} \left[ \begin{bmatrix} 1 + \sqrt{2}i \\ 1 \end{bmatrix} \right],
\]

and so \( \left\{ \begin{bmatrix} 1 + \sqrt{2}i \\ 1 \end{bmatrix} \right\} \) is a basis for the eigenspace \( E_{1+\sqrt{2}i} \).

To find the eigenspace \( E_{1-\sqrt{2}i} \), we need to find the null space of \( A - (1 - \sqrt{2}i)I \). We row reduce the associated augmented matrix:

\[
\begin{bmatrix}
1 & 0 & 3 \\
5 & 1 & 1 \\
0 & 1 & 2
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{bmatrix}
\]

The solution set is \( E_{1-\sqrt{2}i} \). We see that

\[
E_{1-\sqrt{2}i} = \text{span} \left[ \begin{bmatrix} 1 - \sqrt{2}i \\ -1 \end{bmatrix} \right],
\]

and so \( \left\{ \begin{bmatrix} 1 - \sqrt{2}i \\ -1 \end{bmatrix} \right\} \) is a basis for the eigenspace \( E_{1-\sqrt{2}i} \).

Section 4.2:

(1) Using cofactor expansion along the first row, we have

\[
\begin{vmatrix}
1 & 0 & 3 \\
5 & 1 & 1 \\
0 & 1 & 2
\end{vmatrix} = 1 \begin{vmatrix}
1 & 1 \\
0 & 2
\end{vmatrix} - 0 + 3 \begin{vmatrix}
1 & 1 \\
0 & 2
\end{vmatrix}
\]

\[
= 1(2 - 1) - 0 + 3(5 - 0)
\]

\[
= 16
\]
Using cofactor expansion along the first column, we have

\[
\begin{vmatrix}
1 & 0 & 3 \\
5 & 1 & 1 \\
0 & 1 & 2
\end{vmatrix}
= 1 \begin{vmatrix}
1 & 1 \\
1 & 2
\end{vmatrix} - 5 \begin{vmatrix}
0 & 3 \\
1 & 2
\end{vmatrix} + 0 \begin{vmatrix}
0 & 3 \\
1 & 1
\end{vmatrix}
= 1(2 - 1) - 5(0 - 3) + 0
= 16
\]

(6) Using cofactor expansion along the first row, we have

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix}
= 1 \begin{vmatrix}
5 & 6 \\
8 & 9
\end{vmatrix} - 2 \begin{vmatrix}
4 & 6 \\
7 & 9
\end{vmatrix} + 3 \begin{vmatrix}
4 & 5 \\
7 & 8
\end{vmatrix}
= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)
= 0
\]

Using cofactor expansion along the first column, we have

\[
\begin{vmatrix}
1 & 2 & 3 \\
4 & 5 & 6 \\
7 & 8 & 9
\end{vmatrix}
= 1 \begin{vmatrix}
5 & 6 \\
8 & 9
\end{vmatrix} - 4 \begin{vmatrix}
2 & 3 \\
8 & 9
\end{vmatrix} + 7 \begin{vmatrix}
2 & 3 \\
8 & 6
\end{vmatrix}
= 1(45 - 48) - 4(18 - 24) + 7(12 - 15)
= 0
\]

(9) Using cofactor expansion along the third row, we have

\[
\begin{vmatrix}
-4 & 1 & 3 \\
2 & -2 & 4 \\
1 & -1 & 0
\end{vmatrix}
= 1 \begin{vmatrix}
1 & 3 \\
-2 & 4
\end{vmatrix} - (-1) \begin{vmatrix}
-4 & 3 \\
2 & 4
\end{vmatrix} + 0 \begin{vmatrix}
-4 & 1 \\
2 & -2
\end{vmatrix}
= 1(4 + 6) + 1(-16 - 6) + 0
= -12
\]

7
(13) Starting with cofactor expansion along the third row, we have
\[
\begin{vmatrix}
1 & -1 & 0 & 3 \\
2 & 5 & 2 & 6 \\
0 & 1 & 0 & 0 \\
1 & 4 & 2 & 1 \\
\end{vmatrix} = -1 \begin{vmatrix}
1 & 0 & 3 \\
2 & 2 & 6 \\
1 & 2 & 1 \\
\end{vmatrix}
\]
\[
= -1 \left( 1 \begin{vmatrix} 2 & 6 \\ 2 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 6 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 2 & 2 \\ 1 & 2 \end{vmatrix} \right)
\]
\[
= -1[(2 - 12) - 0 + 3(4 - 2)]
\]
\[
= -1(-4)
\]
\[
= 4
\]

(23) We row reduce the given matrix \(A\) keeping track of our row operations along the way:
\[
A = \begin{bmatrix}
-4 & 1 & 3 \\
2 & -2 & 4 \\
1 & -1 & 0 \\
\end{bmatrix} \rightarrow B = \begin{bmatrix}
1 & -1 & 0 \\
2 & -2 & 4 \\
-4 & 1 & 3 \\
\end{bmatrix}
\]
\[
\rightarrow C = \begin{bmatrix}
1 & -1 & 0 \\
0 & 0 & 4 \\
0 & -3 & 3 \\
\end{bmatrix}
\]
\[
\rightarrow D = \begin{bmatrix}
1 & -1 & 0 \\
0 & -3 & 3 \\
0 & 0 & 4 \\
\end{bmatrix}
\]

By Theorem 4.2, we have:
\[
\det(B) = -\det(A); \det(C) = \det(B); \det(D) = -\det(C).
\]
So,
\[
\det(A) = -\det(B) = -\det(C) = \det(D) = 1 \begin{vmatrix} -3 & 3 \\ 0 & 4 \end{vmatrix} = -12.
\]

(26) We can row reduce the given matrix \(A\) with the operation \(R_3 \rightarrow R_3 - 2R_1\). This gives a new matrix \(B\) such that \(\det(B) = \det(A)\). Moreover, since the third row of \(B\) is a zero row, \(\det(B) = 0\). We conclude that \(\det(A) = 0\).
(35) By Theorem 4.3,
\[
\begin{vmatrix}
2a & 2b & 2c \\
d & e & f \\
g & h & i \\
\end{vmatrix} = 2
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix} = 2(4) = 8.
\]

(36) By Theorem 4.10,
\[
\begin{vmatrix}
a & d & g \\
b & e & h \\
c & f & i \\
\end{vmatrix} = 4.
\]
So, by Theorem 4.3,
\[
\begin{vmatrix}
3a & 3d & 3g \\
-b & -e & -h \\
2c & 2f & 2i \\
\end{vmatrix} = 4(3)(-1)(2) = -24.
\]
We again apply Theorem 4.10 to obtain
\[
\begin{vmatrix}
3a & -b & 2c \\
3d & -e & 2f \\
3g & -h & 2i \\
\end{vmatrix} = -24.
\]

(37) By Theorem 4.3,
\[
\begin{vmatrix}
d & e & f \\
a & b & c \\
g & h & i \\
\end{vmatrix} = -
\begin{vmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{vmatrix} = -4.
\]

(40) Observe that we have the following row reductions:
\[
A = \begin{bmatrix}
a & b & c \\
d & e & f \\
g & h & i \\
\end{bmatrix} \rightarrow B = \begin{bmatrix}
a & b & c \\
2d & 2e & 2f \\
g & h & i \\
\end{bmatrix} \rightarrow C = \begin{bmatrix}
a & b & c \\
2d - 3g & 2e - 3h & 2f - 3i \\
g & h & i \\
\end{bmatrix}.
\]
By Theorem 4.3,
\[
\det(C) = \det(B) = 2 \det(A) = 2(4) = 8.
\]

9
(45) Using cofactor expansion along the first column of $A$, we compute

\[
\det(A) = k \begin{vmatrix} (k+1) & 1 \\ -8 & (k-1) \end{vmatrix} - 0 \begin{vmatrix} -k & 3 \\ -8 & (k-1) \end{vmatrix} + k \begin{vmatrix} -k & 3 \\ (k+1) & 1 \end{vmatrix} \\
= k(k^2 + 7) + k(-4k - 3) \\
= k^3 - 4k^2 + 4k \\
= k(k^2 - 4k + 4) \\
= k(k - 2)^2
\]

By Theorem 4.6, $A$ is invertible if and only if $\det(A) \neq 0$. We conclude that $A$ is invertible if and only if $k \neq 0, 2$. 