Problem Set 2 Solutions
Due: Thursday, March 25

(1) Fix $\mathcal{H} := (1, 4, 6, 9, 10, 13, 13, \ldots)$ and let $S := k[x_1, x_2, x_3, x_4]$ where $k$ is a field. Does there exist a homogeneous ideal $I \subset S$ such that $H(S/I) = \mathcal{H}$? Provide two reasons for your answer: one using an O-sequence approach and one using an order ideal of monomials approach.

Solution: Denote the $i$th component of $\mathcal{H}$ by $d_i$ for $i \geq 0$.
- **O-Sequence Approach**: In order for $\mathcal{H} = H(S/I)$ for some homogeneous ideal $I$ we need $d_{i+1} \leq d_i^{<i>}$ for $i \geq 1$ and $d_0 = 1$. Consider $i = 4$. We calculate the 4-binomial expansion of $d_4 = 10$ to be:

$$10 = \left(\begin{array}{c} 5 \\ 4 \end{array}\right) + \left(\begin{array}{c} 4 \\ 3 \end{array}\right) + \left(\begin{array}{c} 2 \\ 2 \end{array}\right).$$

Thus, $10^{<4>} = \left(\begin{array}{c} 6 \\ 5 \end{array}\right) + \left(\begin{array}{c} 5 \\ 4 \end{array}\right) + \left(\begin{array}{c} 3 \\ 3 \end{array}\right) = 6 + 5 + 1 = 12.

But $d_5 = 13 > 10^{<4>} = 12$ and so $\mathcal{H}$ is not an O-sequence. By Macaulay’s Theorem we conclude that there is no homogeneous ideal $I \subset S$ such that $H(S/I) = \mathcal{H}$.

- **Order Ideal of Monomials Approach**: For what follows we use the degree reverse lexicographic ordering with $x_1 > x_2 > \cdots > x_n$. Let $\mathcal{M} = \cup_{t \geq 0} \mathcal{M}_t$, where $\mathcal{M}_t$ is the set of $d_t$ largest monomials of degree $t$. Setting $t = 4$ and $t = 5$ we find

$$\mathcal{M}_4 = \{x_1^4, x_1^3x_2, x_1^2x_2^2, x_1x_2^3, x_1^4, x_1^3x_3, x_1^2x_2x_3, x_1x_2^2x_3, x_2^3x_3, x_1^2x_3^2\}$$

and

$$\mathcal{M}_5 = \{x_1^5, x_1^4x_2, x_1^3x_2^2, x_1x_2^3, x_2^4, x_1^4x_3, x_1^3x_2x_3, x_1^2x_2^2x_3, x_1x_2^2x_3, x_2^3x_3, x_1^3x_3^2, x_1^2x_2x_3^2\}.$$  

Note that $x_1x_2x_3^2$ is of degree 4 and divides $x_1^2x_2x_3^2 \in \mathcal{M}_5$. However, $x_1x_2x_3^2 \notin \mathcal{M}_4$. We conclude that $\mathcal{M}$ is not an order ideal of monomials. So, by Macaulay’s Theorem, there does not exist a homogeneous ideal $I \subset S$ with $H(S/I) = \mathcal{H}$.

(2) For this exercise we use the same notation that was set up in our discussion of lifting monomial ideals. Let $f = x^\alpha \in S = k[x_1, \ldots, x_n]$. Prove the following two facts:

(a) $\overline{f} = 0$ if and only if $\alpha \preceq \beta$;

Proof. Let $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$. Recall that

$$\overline{f} = \prod_{j=1}^{n} \prod_{i=0}^{a_j-1} (x_j - t_{j,i}x_0)$$

and

$$\overline{\beta} = [1 : t_{1,b_1} : t_{2,b_2} : \cdots : t_{n,b_n}].$$

Using the fact that the $t_{l,m}$ are chosen to be distinct for a fixed $l$, we see that

$\overline{f} = 0 \iff t_{j,b_j} = t_{j,i}$ for some $1 \leq j \leq n$ and some $0 \leq i \leq a_j - 1$

$\iff b_j = i$ for some $1 \leq j \leq n$ and some $0 \leq i \leq a_j - 1$

$\iff b_j < a_j$ for some $1 \leq j \leq n$

$\iff \alpha \preceq \beta$. 

□
(3) In this exercise we further explore Hilbert functions of distinct points in projective 2-space. Let \( S = k[x_1, x_2] \), where \( k \) is an algebraically closed field of characteristic zero. Further, let \( J \subseteq S \) be a homogeneous ideal such that \( \sqrt{J} = (x_1, x_2) \). We set \( \alpha(J) \) to be the least degree of a non-zero homogeneous polynomial in \( J \).

(a) Set \( B = S/J \). Prove that

\[
H(B, t) = \begin{cases} 
  t + 1 & \text{for } t < \alpha(J) \\
  \leq \alpha(J) & \text{for } t \geq \alpha(J) 
\end{cases}
\]

**Proof.** First observe that

\[
H(B, t) = \dim_k(S_t) - \dim_k(J_t) = (t + 1) - \dim_k(J_t).
\]

For \( t < \alpha(J) \), we have that \( J_t = 0 \) and so \( H(B, t) = t + 1 \). For the second part of the claim, note that there exists a non-zero element \( F \in J \) such that \( \deg(F) = \alpha(J) \). Hence, \( \dim_k(J_{\alpha(J)}) \geq 1 \) and so \( H(B, \alpha(J)) \leq (\alpha(J) + 1) - 1 = \alpha(J) \). Finally, fix \( t > \alpha(J) \). We have that \( (F) \subseteq J \) and so

\[
\dim_k(S_{t-\alpha(J)}) = \dim_k((F)_t) \leq \dim_k(J_t).
\]

Therefore,

\[
H(B, t) = \dim_k(S_t) - \dim_k(J_t) \\
\leq \dim_k(S_t) - \dim_k((F)_t) \\
= \dim_k(S_t) - \dim_k(S_{t-\alpha(J)}) \\
= (t + 1) - (t - \alpha(J) + 1) \\
= \alpha(J).
\]

\( \square \)

(b) Let \( V \subseteq S_t \) be a non-zero subspace of \( S_t \). Denote by \( S_1 V \) the subspace of \( S_{t+1} \) generated by \( \{Lv \mid L \in S_1 \text{ and } v \in V\} \). Prove that

\[
\dim_k(S_1 V) \geq (\dim_k V) + 1.
\]

**Proof.** Let \( F_1, \ldots, F_l \) be a basis for \( V \). It is clear that \( \{x_1 F_1, \ldots, x_1 F_l, x_2 F_1, \ldots, x_2 F_l\} \) spans \( S_1 V \) and \( \{x_1 F_1, \ldots, x_1 F_l\} \) is a linearly independent set. We will be done if we can show that \( x_2 F_j \) is not in the span of \( \{x_1 F_1, \ldots, x_1 F_l\} \) for some \( j \).

For \( 1 \leq i \leq l \) write \( F_i = x_1^q F_i' \), where \( q \) is chosen so that \( x_1 \) does not divide some \( F_i' \) (\( q = 0 \) is possible). We claim that \( x_2 F_j \) is not in the span of \( \{x_1 F_1, \ldots, x_1 F_l\} \). To see this, assume to the contrary that we can find constants \( c_1, \ldots, c_l \in k \) such that

\[
\sum_{i=1}^l c_i x_1 F_i = x_2 F_j.
\]
Then
\[ x_1^{q+1} \sum_{i=1}^l c_i F'_i = x_2 x_1^q F'_j \]
which implies that \( x_1 \) divides \( F'_j \), a contradiction. \( \square \)

(c) Let \( X = \{P_1, \ldots, P_l\} \) be a set of distinct points in \( \mathbb{P}^2 \). We set \( \alpha = \alpha(X) \) to be the least degree of a non-zero homogeneous polynomial in \( I(X) \). Show that \( \Delta H(X) \) has the form
\[ \Delta H(X) = \{1, 2, 3, \ldots, \alpha - 1, \alpha, \Delta H(X, \alpha), \Delta H(X, \alpha + 1), \ldots \} \]
where \( \alpha \geq \Delta H(X, \alpha) \geq \Delta H(X, \alpha + 1) \geq \Delta H(X, \alpha + 2) \geq \cdots \).

\textbf{Proof.} Let \( R = k[x_0, x_1, x_2] \) and \( I = I(X) \). Since \( X \) is a finite set of distinct points there is a linear form which misses \( X \) entirely. After a linear change of variables, we may assume this linear form is \( x_0 = 0 \). Then \( x_0 \) is a non-zero-divisor in \( A = R/I \). Note that \( R/(I, x_0) \cong S/J \) where \( J \) the homogeneous ideal obtained by setting \( x_0 = 0 \) in the generators of \( I \). In addition, \( \sqrt{J} = (x_1, x_2) \) and \( H(S/J) = \Delta H(X) \). Part (a) can be applied to see that \( \Delta H(X, t) = t + 1 \) for \( t < \alpha \) and \( \Delta H(X, t) \leq \alpha \) for \( t \geq \alpha \).

To see that \( \Delta H(X, \alpha + t) \geq \Delta H(X, \alpha + t + 1) \) for all \( t \geq 0 \), note that \( S J_{\alpha+t} \subseteq J_{\alpha+t+1} \).

Thus, by part (b), \( \dim_k (J_{\alpha+t+1}) \geq \dim_k (J_{\alpha+t}) + 1 \). Therefore,
\[
\begin{align*}
\dim_k (J_{\alpha+t+1}) &\geq \dim_k (J_{\alpha+t}) + 1 \\
\Rightarrow \alpha + t + 1 - \dim_k (J_{\alpha+t}) &\geq \alpha + t + 2 - \dim_k (J_{\alpha+t+1}) \\
\Rightarrow \dim_k (S_{\alpha+t}) - \dim_k (J_{\alpha+t}) &\geq \dim_k (S_{\alpha+t+1}) - \dim_k (J_{\alpha+t+1}) \\
\Rightarrow H(S/J, \alpha + t) &\geq H(S/J, \alpha + t + 1) \\
\Rightarrow \Delta H(X, \alpha + t) &\geq \Delta H(X, \alpha + t + 1). \\
\end{align*}
\]
\( \square \)

(4) Find all possible Hilbert functions for 9 distinct points in \( \mathbb{P}^2 \). Pick one of the Hilbert functions \( H \) and find a set \( X \subset \mathbb{P}^2 \) of 9 distinct points in \( \mathbb{P}^2 \) such that \( H(X) = H \). How do you know that the constructed set of points has the selected Hilbert function?

\textbf{Solution:} There are 8 possible Hilbert functions for 9 distinct points in \( \mathbb{P}^2 \). The possible sequences are listed in the following table:

<table>
<thead>
<tr>
<th></th>
<th>( H(X) )</th>
<th>( \Delta H(X) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>(1, 2, 3, 4, 5, 6, 7, 8, 9, 9, \ldots)</td>
<td>(1, 1, 1, 1, 1, 1, 1, 1, 0, 0, \ldots)</td>
</tr>
<tr>
<td>B</td>
<td>(1, 3, 4, 5, 6, 7, 8, 9, 9, \ldots)</td>
<td>(1, 2, 1, 1, 1, 1, 1, 0, 0, \ldots)</td>
</tr>
<tr>
<td>C</td>
<td>(1, 3, 5, 6, 7, 8, 9, 9, \ldots)</td>
<td>(1, 2, 2, 1, 1, 1, 0, 0, \ldots)</td>
</tr>
<tr>
<td>D</td>
<td>(1, 3, 5, 7, 8, 9, 9, \ldots)</td>
<td>(1, 2, 2, 1, 1, 1, 0, 0, \ldots)</td>
</tr>
<tr>
<td>E</td>
<td>(1, 3, 5, 7, 9, 9, \ldots)</td>
<td>(1, 2, 2, 2, 2, 0, 0, \ldots)</td>
</tr>
<tr>
<td>F</td>
<td>(1, 3, 6, 7, 8, 9, 9, \ldots)</td>
<td>(1, 2, 3, 1, 1, 1, 0, 0, \ldots)</td>
</tr>
<tr>
<td>G</td>
<td>(1, 3, 6, 8, 9, 9, \ldots)</td>
<td>(1, 2, 3, 2, 1, 0, 0, \ldots)</td>
</tr>
<tr>
<td>H</td>
<td>(1, 3, 6, 9, 9, \ldots)</td>
<td>(1, 2, 3, 3, 0, 0, \ldots)</td>
</tr>
</tbody>
</table>

To argue that these are all the possible Hilbert functions, the following facts will be useful:
- \( H(X) \) is a differentiable O-sequence with \( H(X, 1) \leq 3 \)
- \( H(X, d) \leq 9 \) for all \( d \geq 0 \)
- \( H(X, d) = 9 \) for \( d \geq 9 \)
- \( \sum_{t=0}^8 \Delta H(X, t) = 9 \) (this follows immediately from the above facts)
We know that $1 \leq H(\mathcal{X}, 1) \leq 3$ and so $1 \leq \Delta H(\mathcal{X}, 1) \leq 2$. We consider these 2 cases. The details of each case is argued by carefully considering the above facts.

1. Suppose $\Delta H(\mathcal{X}, 1) = 1$. Since $\Delta H(\mathcal{X})$ is an O-sequence, $\Delta H(\mathcal{X}, t) \leq 1$ for $t \geq 2$. The only possible sequence for $H(\mathcal{X})$ is the sequence in Case A.

2. Suppose $\Delta H(\mathcal{X}, 1) = 2$. Then, since $\Delta H(\mathcal{X})$ is an O-sequence, $1 \leq \Delta H(\mathcal{X}, 2) \leq 3$. This leads to 3 possible situations.
   
   (i) If $\Delta H(\mathcal{X}, 2) = 1$ then $\Delta H(\mathcal{X}, t) \leq 1$ for $t \geq 2$. This gives Case B.
   
   (ii) If $\Delta H(\mathcal{X}, 2) = 2$ then $1 \leq \Delta H(\mathcal{X}, 3) \leq 2$. If $\Delta H(\mathcal{X}, 3) = 1$ then we must be in Case C. If $\Delta H(\mathcal{X}, 3) = 2$ then $1 \leq \Delta H(\mathcal{X}, 4) \leq 2$: if $\Delta H(\mathcal{X}, 4) = 1$ then we must be in Case D; if $\Delta H(\mathcal{X}, 4) = 2$ then we are in Case E.
   
   (iii) If $\Delta H(\mathcal{X}, 2) = 3$ then $1 \leq \Delta H(\mathcal{X}, 3) \leq 4$. If $\Delta H(\mathcal{X}, 3) = 1$ then we must be in Case F. If $\Delta H(\mathcal{X}, 3) = 2$ then $\Delta H(\mathcal{X}, 4) = 1$ giving Case G. If $\Delta H(\mathcal{X}, 3) = 3$ then $\Delta H(\mathcal{X}, 4) = 0$ giving Case H. We can never have $\Delta H(\mathcal{X}, 3) = 4$ since $\sum_{t=0}^8 \Delta H(\mathcal{X}, t) = 9$.

We now concentrate on $H = (1, 3, 5, 7, 8, 9, \ldots)$. Using the method of lifting monomial ideals with $f_{j,i} = i$ gives $H(\mathcal{X}) = H$ where

$\mathcal{X} = \{(1 : 0 : 0), (1 : 1 : 0), (1 : 0 : 1), (1 : 2 : 0), (1 : 1 : 1), (1 : 3 : 0), (1 : 2 : 1), (1 : 4 : 0), (1 : 5 : 0)\}$.

(5) Suppose that $I$ is a homogeneous ideal in the ring $R = k[x_0, \ldots, x_n]$ where $k$ is an algebraically closed field of characteristic 0. Suppose that $I_d \neq 0$ and that $H(R/I)$ has maximal growth in degree $d$. Prove that $I_d$ and $I_{d+1}$ have a greatest common divisor of positive degree in the following two cases:

(a) $n = 1$ and $H(R/I, d) \geq 1$;

Proof. From our discussion on maximal growth of Hilbert functions it suffices to demonstrate that $PGCD(I_d)$ is positive. We have

$$PGCD(I_d) = \max\{j \mid f_{1,j}(d) \leq H(R/I, d)\},$$

where by definition

$$f_{1,j} = \binom{d+1}{1} - \binom{d-j+1}{1} = (d+1) - (d-j+1) = j.$$

Setting $j = 1$, we see that $1 = f_{1,1} \leq H(R/I, d)$. Hence $PGCD(I_d) > 0$ and we are done. \qed

(b) $n = 2$ and $H(R/I, d) \geq d + 1$.

Proof. As in part (a), it suffices to demonstrate that $PGCD(I_d)$ is positive. We have

$$PGCD(I_d) = \max\{j \mid f_{2,j}(d) \leq H(R/I, d)\},$$

where

$$f_{2,j} = \binom{d+2}{2} - \binom{d-j+2}{2}.$$  

Setting $j = 1$, we see that

$$f_{2,1} = \binom{d+2}{2} - \binom{d+1}{2} = d + 1 \leq H(R/I, d).$$

We conclude that $PGCD(I_d) > 0$ which completes the proof. \qed