Available online at www.sciencedirect.com
ScienceDirect
Journal of Pure and Applied Algebra 212 (2008) 122–131
JOURNAL OF PURE AND APPLIED ALGEBRA
www.elsevier.com/locate/jpaa

Growth conditions for a family of ideals containing regular sequences

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Received 31 May 2006; received in revised form 11 April 2007; accepted 4 May 2007
Available online 24 May 2007

Communicated by C.A. Weibel

Abstract

It has been conjectured by Eisenbud–Green–Harris that *lex-plus-powers ideals* exhibit extremal conditions among all homogeneous ideals containing a regular sequence of forms in fixed degrees. In the same spirit, we consider a family of homogeneous ideals in $k[x, y, z]$ which contain a regular sequence of forms $F, G \in k[x, y]$ and compare the growth of these ideals with special monomial ideals sharing similar properties.

MSC: 13C40; 13D40

1. Introduction

Let R be the polynomial ring $k[x_1, \dots, x_n]$ in n variables, where k is an algebraically closed field of characteristic zero and $\deg(x_i) = 1$. A well-known theorem of Macaulay characterizes the sequences which occur as the Hilbert function of any k-algebra R/I, where I is a homogeneous ideal [14]. Much effort has gone into generalizing Macaulay's Theorem. In particular, Clements–Lindström [3] and Greene–Kleitman [13] give results which can be used to obtain a lower bound for $\dim_k(A_1 J_d)$ where J is a monomial ideal in $A := R/(x_1^{a_1}, x_2^{a_2}, \dots, x_n^{a_n})$. Cooper–Roberts [4,5] extend these results to include non-monomial ideals in A.

One might wonder if there is a "Macaulay-type" characterization for the Hilbert functions of k-algebras R/I where I is any homogeneous ideal containing a regular sequence in fixed degrees. This question brings us to the *Lex-Plus-Powers Conjecture* which, if true, implies that the growth bounds of Clements–Lindström also characterize such Hilbert functions. More precisely, let $1 \le a_1 \le \cdots \le a_n$ be integers. A *lex-plus-powers ideal* is a monomial ideal $L \subseteq R$ which is minimally generated by $\{x_1^{a_1}, \dots, x_n^{a_n}, m_1, \dots, m_r\}$, where, for $j = 1, \dots, r$, all monomials of degree $\deg(m_j)$ which are larger than m_j in the degree-lexicographic ordering are in L. Motivated by the *Cayley–Bacharach Property*, Eisenbud–Green–Harris conjectured that if $I \subseteq R$ is an ideal containing a regular sequence in degrees $a_1, \dots, a_n$ and L is a lex-plus-powers ideal such that $\dim_k(I_d) = \dim_k(L_d)$ for some d, then $\dim_k(R_1 I_d) \ge \dim_k(R_1 L_d)$ [8,9]. This conjecture is called the *Lex-Plus-Powers Conjecture for Hilbert Functions*, denoted LPPH or EGH [11,15].

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0022-4049/$ - see front matter
doi:10.1016/j.jpaa.2007.05.008

The LPPH is only known to be true in some exceptional cases. The conjecture has been proven in the cases where L is an *almost complete intersection* [10], and where I is a monomial ideal containing $x_1^{a_1}, \dots, x_n^{a_n}$ [3]. In [4,5] there is a proof of the conjecture for arbitrary ideals I containing $x_1^{a_1}, \dots, x_n^{a_n}$. In addition, the conjecture is known to be true when $n = 2$ [15]. The EGH Conjecture was originally stated in the case when each $a_i = 2$: in this case, Richert [15] has verified the LPPH for $n \leq 5$. Recently, Caviglia–Maclagan have announced some other known cases [2]. On the geometric side, Cooper [4] has proven the LPPH for Artinian reductions of ideals of distinct points in $\mathbb{P}^2$, as well as in $\mathbb{P}^3$ under some assumptions on the degrees a_1, a_2, a_3.

In this paper we fix $R := k[x, y, z]$ and $S := k[x, y]$, where $x >_{d\text{-}lex} y >_{d\text{-}lex} z$ (here "*d-lex*" denotes the degree-lexicographic ordering). We also fix $I := (F, G, H_1, H_2, \dots, H_t) \subseteq R$ to be a homogeneous ideal such that:

(1) I is minimally generated by $F, G, H_1, H_2, \dots, H_t$;
(2) $\deg(F) = \deg(G) = \deg(H_1) = \deg(H_2) = \cdots = \deg(H_t) = d$;
(3) $F, G \in S$ is a regular sequence of forms.

Fix $J \subseteq R$ to be the ideal generated by x^d, y^d and the t *largest* monomials, with respect to the degree-lexicographic ordering, in $R_d \setminus \{x^d, y^d\}$.

Main Theorem. $\dim_k(R_1 I_d) \geq \dim_k(R_1 J_d)$.

This paper is organized as follows. Section 2 gives the background: we discuss consequences of maximal growth of Hilbert functions and special deformations of R. In Section 3 we study the growth of the ideal J and verify the inequality $\dim_k(R_1 I_d) \geq \dim_k(R_1 J_d)$ in special cases. We then prove the bound in general, and in Section 4 state a generalization. The results of this paper have been extracted from parts of my Ph.D. dissertation [4].

2. Preliminary definitions and results

We fix k to be an algebraically closed field of characteristic zero.

2.1. Hilbert functions and maximal growth

Fix $R := k[x_0, \dots, x_n]$ with the standard grading. Let $A := R/I$, where I is a homogeneous ideal. The *Hilbert function* of A is the sequence $H(A) = \{H(A, t)\}_{t \geq 0}$, where $H(A, t) := \dim_k(A_t) = \dim_k(R_t/I_t)$. The growth of $H(A)$ is well-known. To describe this explicitly, recall that if $h, i \geq 0$ are integers, then we can uniquely write

$$h = \binom{m_i}{i} + \binom{m_{i-1}}{i-1} + \cdots + \binom{m_j}{j},$$

where $m_i > m_{i-1} > \cdots > m_j \geq j \geq 1$. We define

$$h^{\langle i \rangle} := \binom{m_i + 1}{i + 1} + \binom{m_{i-1} + 1}{i} + \cdots + \binom{m_j + 1}{j + 1}.$$

We will often use the fact that if $i \geq h$ then $h^{\langle i \rangle} = h$.

Theorem 2.1 (*[14,16] Macaulay's Theorem*). *Let $\mathcal{H} := \{c_i\}_{i \geq 0}$ be a sequence of non-negative integers. The following are equivalent:*

(1) *$c_0 = 1$ and $c_{i+1} \leq c_i^{\langle i \rangle}$ for all $i \geq 1$ (i.e. $\mathcal{H}$ is an O-sequence);*
(2) *Using the degree reverse-lexicographic ordering, for each $i \geq 0$, let M_i be the largest c_i monomials of R_i. Then $M := \cup_{i \geq 0} M_i$ is an order ideal;*
(3) *$\mathcal{H} = H(R/I)$ for some quotient R/I where I is a homogeneous ideal.*

Let $I = \oplus_{t \geq 0} I_t \subseteq R$ be a homogeneous ideal. Macaulay's Theorem implies that we can estimate $\dim_k(R_1 I_d)$ with a *lex-segment ideal*. That is, if L is the ideal generated by the $\dim_k(I_d)$ largest monomials of R_d with respect to the degree-lexicographic ordering, then $\dim_k(R_1 I_d) \geq \dim_k(R_1 L_d)$.

Bigatti–Geramita–Migliore [1] study Hilbert functions having maximal growth.

Definition 2.2. Let $I \subseteq R$ be a homogeneous ideal. If $b_i = H(R/I, i)$ for $i \geq 0$, then we say that $H(R/I)$ has *maximal growth in degree d* if $b_{d+1} = b_d^{\langle d \rangle}$.

The results of [1] which are of particular interest to us are those concerning greatest common divisors (GCD). For $r \geq 1$, $s \geq 1$, and $x \geq s$, we define $f_{r,s}(x) := \binom{x+r}{r} - \binom{x-s+r}{r}$. Also, $f_{r,0} := 0$ for all r and all x.

Definition 2.3. Let $I \subseteq R$ be a homogeneous ideal such that $I_d \neq 0$. The *potential GCD of* I_d, denoted PGCD, is $\max\{s \mid f_{n,s}(d) \leq H(R/I, d)\}$.

Note that the PGCD is the largest degree *possible* for a common divisor of I_d.

Proposition 2.4 (*[1, Proposition 2.7]*). *Let $I \subseteq R$ be a homogeneous ideal such that $I_d \neq 0$. Assume that I_d has PGCD $= s > 0$ and that $H(R/I)$ has maximal growth in degree d. Then both I_d and I_{d+1} have a GCD of degree s.*

Corollary 2.5. *Let $I \subseteq S = k[x_0, x_1]$ be a homogeneous ideal. Suppose $I_d \neq 0$ and $H(S/I, d) \geq 1$. If $H(S/I)$ has maximal growth in degree d, then I_d and I_{d+1} have a greatest common divisor of positive degree.*

Corollary 2.6. *Let $I \subseteq S = k[x_0, x_1, x_2]$ be a homogeneous ideal such that $I_d \neq 0$ and $H(S/I, d) \geq (d + 1)$. If $H(S/I)$ has maximal growth in degree d, then I_d and I_{d+1} have a greatest common divisor of positive degree.*

2.2. One-parameter torus deformations

Let $R := k[x, y, z]$, $S := k[x, y]$ and $I \subseteq R$ be a homogeneous ideal. Fix integers a_1, a_2, a_3 and let $t \neq 0$ be in k. The *one-parameter torus deformation* is the automorphism ψ_t of R defined by sending $x \mapsto t^{a_1}x$, $y \mapsto t^{a_2}y$ and $z \mapsto t^{a_3}z$. If we treat t as a variable and define $\tilde{I} \subseteq R[t, 1/t]$ to be the ideal generated by $\{\psi_t(f) \mid f \in I\}$, then $R[t, 1/t]/\tilde{I}$ is a flat family of algebras over $k[t, 1/t]$. Thus, as $t \to 0$ there exists a unique limit ideal $\mathscr{I} \subseteq R$ with $\dim_k(\mathscr{I}_l) = \dim_k(I_l)$ for all $l \geq 0$ [7, Chapter 15]. To find $\mathscr{I}$, let $F_1, \ldots, F_l$ be a generating set for I, $\mathscr{J} := (\psi_t(F_1), \psi_t(F_2), \ldots, \psi_t(F_l))$ and consider the union $\bigcup(\mathscr{J} : (t)^d)$ (over all d). If we set $t = 0$, then the resulting ideal equals $\mathscr{I}$ which is the saturation of $\tilde{I}$ in $R[t]$.

Example 2.7. We let $f = x^4 + x^3y$, $g = y^4 - xy^3$, $h = x^2y^2 - xyz^2$, $i = y^3z - x^2z^2 + z^4$ and $I = (f, g, h, i)$. Fix $a_1 = 0, a_2 = 0, a_3 = 1$ so that

$$\psi_t(f) = f, \quad \psi_t(g) = g, \quad \psi_t(h) = x^2y^2 - t^2xyz^2, \quad \psi_t(i) = ty^3z - t^2x^2z^2 + t^4z^4.$$

As $t \to 0$, the “limits” of $\psi_t(f)$, $\psi_t(g)$ and $\psi_t(h)$ are $f_0 = f$, $g_0 = g$, $h_0 = x^2y^2 \in \mathscr{I}$, respectively. In addition, we see that if $t \neq 0$ then

$$\frac{1}{t}\psi_t(i) = y^3z - tx^2z^2 + t^3z^4 \in \tilde{I}.$$

So, as $t \to 0$, we see that the “limit” of $\psi_t(i)$ is $i_0 = y^3z \in \mathscr{I}$.

Let $\mathscr{J} := (\psi_t(f), \psi_t(g), \psi_t(h), \psi_t(i))$ and consider $\bigcup(\mathscr{J} : (t)^d)$. We set $t = 0$ and obtain $\mathscr{I} = (xyz^4 - 1/3y^2z^4, x^2z^4 - y^2z^4, xy^2z^3, y^2z^5, x^3yz^2, y^6, z^7, yz^6, xz^6, xy^3 - y^4, x^4 + x^3y, x^2y^2, y^3z, x^3z^2 + x^2yz^2, x^2yz^3)$.

In Example 2.7, both f, g and f_0, g_0 are regular sequences. This is not always true. For example, let $f = x^3 - 2x^2y$, $g = z^3 + xz^2 - y^3$, $a_1 = 2, a_2 = 1$ and $a_3 = 3$. Then $\psi_t(f) = t^6x^3 - 2t^5x^2y$ and $\psi_t(g) = t^9z^3 + t^8xz^2 - t^3y^3$. Dividing $\psi_t(f)$ by t^5 and $\psi_t(g)$ by t^3 and letting $t \to 0$, we see $f_0 = -2x^2y$ and $g_0 = -y^3$.

Suppose $I \subseteq R$ is a homogeneous ideal that is generated all in degree d. Suppose further that among the generators of I there is a regular sequence $F, G \in k[x, y]$. This is the type of ideal which we will soon focus on. At this point we isolate some observations. Perform a one-parameter torus deformation by sending $x \mapsto x$, $y \mapsto y$ and $z \mapsto tz$ and, as above, let $\mathscr{I}$ be the unique limit ideal obtained as $t \to 0$. We will say that a form $M \in R$ is *evenly divisible by* z^r if every term of M is divisible by z^r and no higher power of z divides M.

Important Observations 2.8. Let I be as above.

(1) We can choose a minimal generating set of I so that if I is generated by $F, G, H_1, \ldots, H_T$, then $\mathscr{I} = (F', G', H_1', H_2', \ldots, H_T', \ldots)$, where F', G', H_l' are the "limits" of F, G, H_l, respectively, and $F', G', H_1', \ldots, H_T'$ form a k-basis for $\mathscr{I}_d$. Of course, $\mathscr{I}$ may have generators in degrees $\geq d+1$.
(2) If a form has terms involving just x and y, then its "limit" is itself, and so $F' = F$ and $G' = G$. If a form has no term involving just x and y, then its "limit" is a form which is evenly divisible by a positive power of z. Hence, every degree d generator of $\mathscr{I}$ is either in $k[x, y]$ or is evenly divisible by a positive power of z. We "order" the degree d generators of $\mathscr{I}$ such that all forms involving only x and y are listed first, and if H_l', H_{l+1}' are evenly divisible by z^r and z^t, respectively, then $r \leq t$. With these properties, finding a basis for $\mathscr{I}_{d+1}$ is easier than for I_{d+1}.

3. Growth conditions for special ideals

We first recall how ideals generated by a regular sequence of forms can grow.

Proposition 3.1 *([12, Proposition 3.14]). Let $F_1, \ldots, F_s \in k[x_0, \ldots, x_n]$ be a regular sequence where $\deg(F_i) = d$, $1 \leq i \leq s$. If $N := (F_1, \ldots, F_s)$, then $\dim_k(N_{d+l}) = s\binom{l+n}{n}$ for $0 \leq l < d$.*

Let m and n be two monomials. We will write $m >_{d\text{-}lex} n$ if m is larger than n with respect to the degree-lexicographic ordering.

Standing Notation 3.2. We fix $R := k[x, y, z]$, $S := k[x, y]$, where $\deg(x) = \deg(y) = \deg(z) = 1$, $x >_{d\text{-}lex} y >_{d\text{-}lex} z$, and k is an algebraically closed field of characteristic 0. Also fix $I := (F, G, H_1, \ldots, H_t) \subseteq R$ to be a homogeneous ideal such that:

(1) I is minimally generated by $F, G, H_1, H_2, \ldots, H_t$;
(2) $\deg(F) = \deg(G) = \deg(H_1) = \deg(H_2) = \cdots = \deg(H_t) = d$;
(3) $F, G \in S$;
(4) F, G is a regular sequence of forms.

In addition, we fix $J \subseteq R$ to be the monomial ideal generated by x^d, y^d and the t *largest* monomials, with respect to the degree-lexicographic ordering, in $R_d \setminus \{x^d, y^d\}$. We will always order the generators of J as $x^d, y^d, m_1, \ldots, m_t$ where $m_i >_{d\text{-}lex} m_{i+1}$, so it makes sense to speak of the "last generator" of J. Clearly $\dim_k(I_d) = \dim_k(J_d)$.

Theorem 3.3. *If I and J are as above, then $\dim_k(R_1 I_d) \geq \dim_k(R_1 J_d)$.*

If I is a *monomial* ideal, then Theorem 3.3 follows from [3, Corollary 1]. Further, Theorem 3.3 is obviously true if $\dim_k(I_d) = 2$ or if $d = 1$. So we assume $d \geq 2$ and $\dim_k(I_d) \geq 3$. We will divide the proof of Theorem 3.3 into several steps. First, we exhibit the case $d = 2$. We then introduce a grouping (Definition 3.6) which will control the organization for the remaining cases.

Lemma 3.4. *If $d = 2$ in Standing Notation 3.2, then Theorem 3.3 is true.*

Proof. By Proposition 3.1, $\dim_k(R_1 I_2) = \dim_k(I_3) \geq 6$.

Case 1: Fix $\dim_k(I_2) = \dim_k(J_2) = 3$. We see that $\dim_k(R_1 J_2) = 7$. If $\dim_k(R_1 I_2) = 6$, then (by Corollary 2.6) I_2 has a GCD of positive degree, a contradiction.

Case 2: Assume that $4 \leq \dim_k(I_2) = i \leq 5$. Then $\dim_k(R_1 J_2) = i + 4$. We have $1 \leq H(R/I, 2) = 6 - i \leq 2$. By Macaulay's Theorem $H(R/I, 3) \leq 6 - i$, i.e. $\dim_k(R_1 I_2) \geq 10 - (6 - i) = i + 4$.

Case 3: Suppose finally that $\dim_k(I_2) = 6$. Then the generators of J and I each form a k-basis for R_2. So, $\dim_k(R_1 J_2) = 10 = \dim_k(R_3) = \dim_k(R_1 I_2)$. □

We now assume $d \geq 3$. It is crucial to understand the growth of the ideal J.

Notation 3.5. Let $V := \langle x^d, y^d, m_1, \ldots, m_t \rangle$ be the k-vector space spanned by x^d, y^d and the t largest monomials of $R_d \setminus \{x^d, y^d\}$, with respect to the degree-lexicographic ordering. We assume $m_i >_{d\text{-}lex} m_{i+1}$ and that $\dim_k(V) \geq 3$.

Definition 3.6. We group the monomials of R_d as follows:

$$\textit{Group } i := \{x^{d-i}y^i, x^{d-i}y^{i-1}z, x^{d-i}y^{i-2}z^2, \dots, x^{d-i}z^i\}, \quad \text{for } 1 \le i \le d-1,$$
$$\textit{Group } 0 := \{x^d, y^d\} \quad \text{and} \quad \textit{Group } d := \{y^{d-1}z, y^{d-2}z^2, \dots, z^d\}.$$

That is, for $1 \le i \le d-1$, Group i is the set of monomials of R_d which are divisible by x^{d-i}, and no higher power of x. It will turn out that $\dim_k(R_1V)$ depends on the group to which the *last basis element* m_t of V belongs.

Observation 3.7. Let V be as in Notation 3.5. Suppose m_t is the jth element of Group i, where $i \ge 1$. Then we can decompose $\dim_k(V)$ as

$$\dim_k(V) = 2 + \sum_{l=2}^{i} l + j.$$

In this sum the first 2 counts x^d, y^d, and each summand in $\sum_{l=2}^{i} l$ counts the basis elements from Group $(l-1)$, where $2 \le l \le i$, and the integer j counts the basis elements of V which are in Group i.

Proposition 3.8. *Let V be as in Observation 3.7.*

(A) *If $i \le d-2$, then* $\dim_k(R_1V) = 6 + \sum_{l=2}^{i}(l+1) + (j+1)$.
(B) *If $i = d-1$, then* $\dim_k(R_1V) = 6 + \sum_{l=2}^{d-1}(l+1) + j$.
(C) *If $i = d$, then* $\dim_k(R_1V) = 6 + \sum_{l=2}^{d-1}(l+1) + d + j$.

Proof. Let $\mathcal{V} := \langle x^{d+1}, y^{d+1}, n_1, \dots, n_w = m_t z\rangle$ be the k-vector space spanned by x^{d+1}, y^{d+1} and the monomials of $R_{d+1} \setminus \{x^{d+1}, y^{d+1}\}$ which are larger than or equal to zm_t, with respect to the degree-lexicographic ordering. We see that $\mathcal{V}$ satisfies the properties of Notation 3.5 using degree $d+1$, where the "last generator" $(m_t z)$ is the $(j+1)$st element of Group $i+1$.

It is straightforward to verify that $R_1V = \mathcal{V} \cup \{xy^d, y^d z\}$. If m_t is in Group i and $1 \le i \le d-2$, then this union is disjoint. If m_t is in Group $(d-1)$, then $\mathcal{V}$ contains the element xy^d but not $y^d z$. If m_t is in Group d, then $\mathcal{V}$ contains xy^d and $y^d z$. We now apply Observation 3.7 to $\mathcal{V}$ to obtain the formulas. □

Note 3.9. Let J be as in Standing Notation 3.2. Using the formulas in Proposition 3.8, notice that when the last element of J passes from the last element of Group i to the first element of Group $(i+1)$, where $0 \le i \le d-3$, then $\dim_k(R_1J_d)$ increases by 2. However, when the last element of J passes from Group i to Group $(i+1)$, for $i = d-2$ or $d-1$, then $\dim_k(R_1J_d)$ increases only by 1. Moreover, when the last element of J passes to the next element within any Group i, for $i \ge 1$, then $\dim_k(R_1J_d)$ increases exactly by 1.

Example 3.10. Below is a table showing the growth of the ideal J for degree $d = 4$. The corresponding numbers in the bottom row indicate the group of Definition 3.6 to which the last generator of J belongs.

$\dim_k(R_1J_4)$:	3	6	8	9	11	12	13	14	15	16	17	18	19	20	21
$\dim_k(J_4)$:	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Group:	0	0	1	1	2	2	2	3	3	3	3	4	4	4	4

We now verify Theorem 3.3 in the case when every generator of I is in $S = k[x, y]$. We begin by obtaining bounds for $\dim_k(R_1I_d)$.

Lemma 3.11. *Suppose $N \subseteq S = k[x, y]$ is a homogeneous ideal whose minimal generators all have degree $d \ge 3$. Suppose further that N_d contains a regular sequence of length two. We have the following.*

(1) *If* $\dim_k(N_d) = d+1$, *then* $\dim_k(R_1N_d) = 2\dim_k(N_d) + 1 = 2d + 3$.
(2) *If* $1 \le \dim_k(N_d) \le d$, *then* $\dim_k(R_1N_d) \ge 2\dim_k(N_d) + 2$.

Proof. Let $M_1, \dots, M_l$ be a minimal generating set for N. Fix $\mathcal{U}$ to be a basis of S_1N_d and $\mathcal{V} := \mathcal{U} \cup \{zM_i\}_{i=1}^{l}$. Clearly the elements of $\mathcal{V}$ form a k-basis for R_1N_d. If $\dim_k(N_d) = d+1$, then $\mathcal{V}$ has $d + 2 + \dim_k(N_d) = 2d + 3$ elements. If $\dim_k(N_d) \le d$ then, by [6, Corollary 2.6], $\dim_k(S_1N_d) \ge 2 + \dim_k(N_d)$, and so $\mathcal{V}$ has at least $2 + \dim_k(N_d) + \dim_k(N_d) = 2 + 2\dim_k(N_d)$ elements. □

Lemma 3.12. *If every generator of I is in S, then Theorem 3.3 is true.*

Proof. If $\dim_k(I_d) = 3$ then, by Lemma 3.11, $\dim_k(R_1 I_d) \geq \dim_k(R_1 J_d) = 8$. So assume $\dim_k(I_d) \geq 4$. Since $\dim_k(J_d) = \dim_k(I_d) \leq d+1 \leq \binom{d+2}{2} - 2d$, the last generator of J is not in Group $(d-1)$ or Group d. Suppose that the last generator of J is the jth element of Group i and decompose $\dim_k(J_d)$ as in Observation 3.7. By Proposition 3.8, $\dim_k(R_1 J_d) = \dim_k(I_d) + 4 + i$. Since $i + 3 \leq \dim_k(I_d)$, we know that $\dim_k(I_d)+4+i \leq 2\dim_k(I_d)+1$. By Lemma 3.11, $\dim_k(R_1 I_d) \geq 2\dim_k(I_d)+1$. Thus, $\dim_k(R_1 I_d) \geq \dim_k(R_1 J_d)$. □

We now give a series of lemmas which, when combined, prove Theorem 3.3.

Lemma 3.13. *If the last generator of J is in Group $(d-1)$, then Theorem 3.3 is true.*

Proof. Suppose the last generator of J is the ith element of Group $(d-1)$. Then $J = (x^d, y^d, x^{d-1}y, \ldots, xy^{d-i}z^{i-1})$ and $\dim_k(J_d) = \binom{d+2}{2} - 2d + i$, where $1 \leq i \leq d$. By Proposition 3.8 and Note 3.9, $\dim_k(R_1 J_d) = \binom{d+3}{2} - 2d + i$.

Case 1: Let $i = d$. By Theorem 2.1, $H(R/I, d+1) = \binom{d+3}{2} - \dim_k(R_1 I_d) \leq d$.

Case 2: Assume $i < d$. By Macaulay's Theorem, $\dim_k(R_1 I_d) \geq \dim_k(R_1 L_d)$, where L_d equals $J_d \setminus \{y^d\}$ with the additional element $xy^{d-i-1}z^i$. We see that $\dim_k(R_1 L_d) = \binom{d+3}{2} - 2d + i - 1$. If $\dim_k(R_1 I_d) = \dim_k(R_1 L_d)$ then, by Corollary 2.6, I_d has a GCD of positive degree, a contradiction. □

Lemma 3.14. *If the last generator of J is in Group d, then Theorem 3.3 is true.*

Proof. By assumption $J = (x^d, y^d, x^{d-1}y, \ldots, xz^{d-1}, y^{d-1}z, \ldots, y^{d-i}z^i)$ and $\dim_k(I_d) = \dim_k(J_d) = \binom{d+2}{2} - d + i$, where $1 \leq i \leq d$. By Proposition 3.8 and Note 3.9, $\dim_k(R_1 J_d) = \binom{d+3}{2} - d + i$. The result follows from the fact that $H(R/I, d+1) = \binom{d+3}{2} - \dim_k(R_1 I_d) \leq (d-i)^{\langle d \rangle} = d - i$. □

Lemma 3.15. *If the last generator of J is the last element of Group i, where $1 \leq i \leq d-2$, then Theorem 3.3 is true.*

Proof. The argument is the same as for Case 2 of Lemma 3.13. □

Theorem 3.16. *If the last generator of J is the first element of Group i, where $1 \leq i \leq d-2$, then Theorem 3.3 is true.*

Proof. For convenience, we postpone the proof until the next section. □

Lemma 3.17. *If the last generator of J is the second element of Group i, where $2 \leq i \leq d-2$, then Theorem 3.3 is true.*

Proof. By Proposition 3.8, $\dim_k(R_1 J_d) = 7 + i + \binom{i+1}{2}$. Let $T := \binom{i+1}{2}$ so that $I = (F, G, H_1, \ldots, H_T, H_{T+1})$. Perform a one-parameter torus deformation by sending $x \mapsto x$, $y \mapsto y$ and $z \mapsto tz$, where $0 \neq t \in k$. Let $\mathscr{I}$ be the limit ideal of I as $t \to 0$. We use the assumptions of Important Observations 2.8.

Define $K := (F, G, H_1', H_2', \ldots, H_T') \subseteq \mathscr{I}$. In the same way that I is associated to J, we associate to K the monomial ideal J'. By Theorem 3.16 and Proposition 3.8, $\dim_k(R_1 K_d) \geq \dim_k(R_1 J_d') = 6 + i + T$. Also, by the assumption on the ordering of the generators of $\mathscr{I}$, zH_{T+1}' is not in any k-basis for $R_1 K_d$. Therefore, $\dim_k(R_1 \mathscr{I}_d) \geq \dim_k(R_1 K_d) + 1 \geq 7 + i + T$, and so

$$\dim_k(R_1 I_d) = \dim_k(I_{d+1}) = \dim_k(\mathscr{I}_{d+1}) \geq \dim_k(R_1 \mathscr{I}_d) \geq 7 + i + T. \quad \square$$

Arguing by induction and repeating the proof of Lemma 3.17 we have:

Lemma 3.18. *If the last generator of J is the jth element of Group i, where $j \neq 1, 2, i+1$ and $3 \leq i \leq d-2$, then Theorem 3.3 is true.*

3.1. A crucial theorem

We now prove Theorem 3.16 with $d \geq 3$, starting with a special case.

Lemma 3.19. *If* $\dim_k(I_d) = \dim_k(J_d) = 3$, *then Theorem* 3.16 *is true.*

Proof. We have $I = (F, G, H)$ and $\dim_k(R_1 J_d) = 8$. By Macaulay's Theorem, $\dim_k(R_1 I_d) \geq \dim_k(R_1 L_d) = 6$, where $L := (x^d, x^{d-1}y, x^{d-1}z) \subseteq R$. If $\dim_k(R_1 I_d) = 6$, then we use the same argument as in Case 2 of the proof of Lemma 3.13 to arrive at a contradiction. So we may as well assume $\dim_k(R_1 I_d) = 7$. Perform a one-parameter torus deformation by sending $x \mapsto x$, $y \mapsto y$ and $z \mapsto tz$. Let $\mathscr{I}$ be the unique limit ideal obtained as $t \to 0$. Recall Important Observations 2.8.

Claim: $\mathscr{I}$ has no generator in degree $d + 1$.

Proof of claim: By assumption, the growth of $H(R/\mathscr{I}) = H(R/I)$ from degree d to degree $(d + 1)$ is exactly one off of the maximal growth allowed by Macaulay's Theorem, and so $\mathscr{I}$ has at most one generator of degree $(d + 1)$. If $\mathscr{I}$ has a generator of degree $(d + 1)$, then $\dim_k(R_1(F, G, H')) = 6$. As we have seen above, this leads to a contradiction, proving the claim.

Thus, $7 = \dim_k(R_1 I_d) = \dim_k(I_{d+1}) = \dim_k(\mathscr{I}_{d+1}) = \dim_k(R_1 \mathscr{I}_d)$, and so in proving Theorem 3.3 in this case for I_d we are reduced to proving it for $\mathscr{I}_d$.

By Lemma 3.12, we may as well suppose that H' is evenly divisible by some positive power of z. It is easy to see that $xF, yF, zF, xG, yG, zG, zH'$ form a k-basis for $R_1\mathscr{I}_d$. Thus, H' is evenly divisible by z^1 and $x\frac{H'}{z}, y\frac{H'}{z}$ are k-linear combinations of F and G. Now let $M := (\frac{H'}{z}, F, G) \subseteq S = k[x, y]$ and note that

$$H(S/M, d - 1) = d - 1 = H(S/M, d).$$

By Corollary 2.5, M_d has a GCD of positive degree, a contradiction. □

Assumption 3.20. We now assume that the last generator of J is the first element of Group i, where $2 \leq i \leq d - 2$ and $d \geq 4$. Thus, by Proposition 3.8,

$$\dim_k(I_d) = \dim_k(J_d) = 2 + \binom{i+1}{2} \quad \text{and} \quad \dim_k(R_1 J_d) = 6 + i + \binom{i+1}{2}.$$

Lemma 3.21. $\dim_k(R_1 I_d) \geq 5 + i + \binom{i+1}{2}$.

Proof. Macaulay's Theorem implies that $\dim_k(R_1 I_d) \geq 4 + i + \binom{i+1}{2}$. The argument from Case 2 of the proof of Lemma 3.13 rules out the case when $\dim_k(R_1 I_d) = 4 + i + \binom{i+1}{2}$. □

Proof of Theorem 3.16. Arguing by contradiction, we assume that $\dim_k(R_1 I_d) = 5 + i + \binom{i+1}{2}$. For ease, fix $T := \binom{i+1}{2}$. Then $I = (F, G, H_1, \ldots, H_T)$, $\dim_k(I_d) = 2 + T$, and $\dim_k(R_1 I_d) = T + 5 + i$. Perform a one-parameter torus deformation by sending $x \mapsto x$, $y \mapsto y$ and $z \mapsto tz$, where $t \neq 0$ is in k. Treat t as a variable and let $\mathscr{I}$ be the unique limit ideal obtained as $t \to 0$. We use the assumptions of Important Observations 2.8. As with Lemma 3.19, $\mathscr{I}$ has no generator of degree $(d + 1)$. So $\dim_k(R_1 I_d) = \dim_k(R_1\mathscr{I}_d) = T + 5 + i$, and we are reduced to proving Theorem 3.16 for $\mathscr{I}_d$. Define $K := (F, G, H'_1, H'_2, \ldots, H'_{T-1}) \subseteq \mathscr{I}$.

Lemma 3.22. $\dim_k(R_1 K_d) = 4 + i + T$.

Proof. Since $R_1 K_d \subseteq R_1\mathscr{I}_d$, we see that $\dim_k(R_1 K_d) \leq 5 + i + T$. By Lemma 3.15, $\dim_k(R_1 K_d) \geq 4 + i + T$. By the assumption on the ordering of $F, G, H'_1, \ldots, H'_T$, we see that zH'_T cannot be written as a k-linear span of elements from any k-basis for $R_1 K_d$. So, $\dim_k(R_1\mathscr{I}_d) \geq \dim_k(R_1 K_d) + 1$. □

Notation-Remark 3.23. The set $\{xF, yG, xG, yG, zF, zG, zH'_1, \ldots, zH'_{T-1}\}$ forms part of a k-basis for $R_1 K_d$. We use the following method to make a k-basis for $R_1 K_d$.

(1) We fix $s \geq 0$ to be the number of generators of K which in addition to F and G are in $S_d = k[x, y]_d$. If $s \geq 1$, we fix t to be the number of additional k-basis elements for $R_1 K_d$ obtained after multiplying $H'_1, \dots, H'_s$ by linear forms in $k[x, y]$. If $s = 0$ we define $t := 0$. This gives us $5 + t + T$ k-basis elements for $R_1 K_d$. We conclude that $t \leq i - 1 < T - 1$.

(2) Let m_w be the number of forms of $\{H'_{s+1}, \dots, H'_{T-1}\}$ which are divisible by z^{l_w}, for $l_w \geq 1$. By assumption, $l_{w+1} \geq l_w + 1$. We assume there are $r \geq 0$ distinct powers of z dividing the forms in $\{H'_{s+1}, \dots, H'_{T-1}\}$. If $r \geq 1$, then we continue constructing a basis for $R_1 K_d$ by adding all the elements in $\{xH'_{s+1}, yH'_{s+1}, \dots, xH'_{s+m_1}, yH'_{s+m_1}\}$ which are not in the k-span of the basis elements of (1). Repeat the present step with $H'_{s+m_1+1}, \dots, H'_{s+m_1+m_2}$ and the new partial basis just found. We continue this process for each m_i, obtaining c_i k-basis elements of $R_1 K'_d$ at each step. If $r \geq 1$, then

$$c_1 + \dots + c_r = i - 1 - t \quad \text{and} \quad m_1 + \dots + m_r = T - 1 - s.$$

If $r = 0$, then $t = i - 1$.

Lemma 3.24. *If $s \geq 1$, then $s + 2 < d + 1$ and $s \leq t$.*

Proof. Since $F, G, H'_1, \dots, H'_s \in k[x, y]_d$ are linearly independent, we see that $s + 2 \leq d + 1$. If $s + 2 = d + 1$, then $(F, G, H'_1, \dots, H'_s)_d = S_d$. Using part (1) of Notation-Remark 3.23, this gives $(d+2)+(T+1) = d+T+3$ k-basis elements for $R_1 K_d$. But $i \leq d - 2$, and so $d + T + 3 \geq i + T + 5$, a contradiction.

We now move to the second part of the claim. Assume that $s > t$. Consider the ideal $A := (F, G, H'_1, \dots, H'_s) \subseteq S$. We have

$$H(S/A, d) = d - 1 - s \quad \text{and} \quad H(S/A, d + 1) = d - 2 - t.$$

By Macaulay's Theorem, $d - 2 - t \leq d - 1 - s$, and so $t < s \leq t + 1$. Hence $s = t + 1$. By Corollary 2.5, A_d has a GCD of positive degree, a contradiction. □

Remark 3.25. By Lemma 3.24, $0 \leq s \leq t \leq i - 1 < T - 1$.

We now consider two lemmas which give bounds for the number of basis elements of K_d which are divisible by a positive power of z.

Lemma 3.26. *Let $2 \leq v \leq r$. If $l_v > l_{v-1} + 1$, then $c_v \geq m_v + 1$. If $l_v = l_{v-1} + 1$, then $c_v \geq m_v - m_{v-1} + 1$.*

Proof. Define the ideal $M \subseteq S = k[x, y]$ as follows: if $l_v > l_{v-1}+1$, then $M := \left(\frac{H'_{s+m_1+\dots+m_{v-1}+1}}{z^{l_v}}, \dots, \frac{H'_{s+m_1+\dots+m_v}}{z^{l_v}}\right)$; if $l_v = l_{v-1} + 1$ and $v > 2$, then $M := \left(\frac{H'_{s+m_1+\dots+m_{v-1}+1}}{z^{l_v}}, \dots, \frac{H'_{s+m_1+\dots+m_v}}{z^{l_v}}, \frac{zH'_{s+m_1+\dots+m_{v-2}+1}}{z^{l_v}}, \dots, \frac{zH'_{s+m_1+\dots+m_{v-1}}}{z^{l_v}}\right)$; if $l_v = l_{v-1} + 1$, $v = 2$ then $M := \left(\frac{H'_{s+m_1+m_2+1}}{z^{l_2}}, \dots, \frac{H'_{s+m_1+m_2}}{z^{l_2}}, \frac{zH'_{s+1}}{z^{l_2}}, \dots, \frac{zH'_{s+m_1}}{z^{l_2}}\right)$.

Now use Notation-Remark 3.23 to find $H(S/M)$. We see that $H(S/M, d - l_v) = d - l_v + 1 - m_v$. If $l_v > l_{v-1} + 1$, then $H(S/M, d - l_v + 1) = d - l_v + 2 - c_v$. If $l_v = l_{v-1} + 1$, then $H(S/M, d - l_v + 1) = d - l_v + 2 - m_{v-1} - c_v$. Applying Macaulay's Theorem completes the argument. □

Lemma 3.27. *$c_1 + c_2 + \dots + c_p \geq m_p - s + (p - 1)$ for $1 \leq p \leq r$.*

Proof. First assume that $l_1 = 1$. We argue by induction. If $s \geq 1$, let $M := \left(F, G, H'_1, \dots, H'_s, \frac{H'_{s+1}}{z}, \dots, \frac{H'_{s+m_1}}{z}\right) \subseteq S = k[x, y]$; and if $s = 0$, let $M := \left(F, G, \frac{H'_1}{z}, \dots, \frac{H'_{m_1}}{z}\right) \subseteq S = k[x, y]$. Using Notation-Remark 3.23, we see that $H(S/M, d - 1) = d - m_1$ and $H(S/M, d) = d + 1 - (s + 2 + c_1)$. By Macaulay's Theorem, $c_1 \geq m_1 - s - 1$.

Assume for a moment that $c_1 = m_1 - s - 1$. Then $H(S/M)$ has maximal growth from degree $(d - 1)$ to degree d. Moreover, the PGCD (in S) of M_{d-1} is easily calculated to be $d - m_1 > 0$. Thus, by Proposition 2.4, M_d has a GCD of positive degree, a contradiction.

Now suppose that we have proved the claim for $p-1$ and consider p. By Lemma 3.26 and our inductive hypothesis,

$$(c_1 + \dots + c_{p-1}) + c_p \geq m_{p-1} - s + (p - 2) + m_p - m_{p-1} + 1 = m_p - s + (p - 1).$$

The case when $l_1 \geq 2$ is argued similarly but with $M := \left(\frac{H'_{s+1}}{z^{l_1}}, \ldots, \frac{H'_{s+m_1}}{z^{l_1}}\right)$. □

Remark 3.28. By Lemma 3.27, for $1 \leq p \leq r$, we have

$$1 \leq m_p \leq (c_1 + \cdots + c_p) + s - p + 1 \leq (i - 1 - t) + s - p + 1 \leq i - p.$$

In particular, $1 \leq m_r \leq i - r$ and so $i \geq 1 + r$.

We now complete the proof of Theorem 3.16. Recall Notation-Remark 3.23.

Case 1: Assume $r \geq 1$. Then

$$T - 1 \leq (i - 1) + (i - 1) + (i - 2) + \cdots + (i - r) = (r + 1)i - \sum_{l=1}^{r} l - 1.$$

This implies $i^2 + i - 2ri - 2i + r^2 + r = (i - r)^2 + r - i \leq 0$. If $r = i - p$ where $p \geq 2$, then $(i - r)^2 + r - i = p^2 - p = p(p - 1) \geq 2$, a contradiction. If $r = i - 1$, then $1 \leq m_r \leq i - r = 1$ and so $m_r = 1$ implying that both xH'_T and yH'_T must be constant multiples of zH'_{T-1}, a contradiction.

Case 2: Now assume $r = 0$. Then all the generators of K_d are in $S = k[x, y]$. Thus, $T + 1 \leq d + 1$. If $T + 1 = d + 1$, then we can take the degree d generators of K to be the $(d + 1)$ k-linearly independent monomials of S_d. Thus,

$$\dim_k(R_1 K_d) = (d + 2) + (T + 1) = d + T + 3 \geq i + 2 + T + 3 = T + i + 5,$$

a contradiction.

Now suppose that $T + 1 < d + 1$. Then $H(S/(F, G, H'_1, \ldots, H'_{T-1}), d) = d - T$ and $H(S/(F, G, H'_1, \ldots, H'_{T-1}), d + 1) = d - i - 1$. By Macaulay's Theorem, we must have $\binom{i+1}{2} = T \leq i + 1$. If $i \geq 3$ this is a clear contradiction. If $i = 2$, then $t = 1$ and (by Remark 3.25) K_d has only 3 elements. But, if $i = 2$, then the last generator of J is the first element of Group 2, and so K_d has 4 elements, a contradiction. □

4. Further growth bounds

We now briefly describe one generalization of Theorem 3.3.

Lemma 4.1. *Let $I \subseteq S = k[x_1, \ldots, x_n]$ be a homogeneous ideal containing a regular sequence of forms $F_1, \ldots, F_n$. We can find general linear forms $L_1, L_2, \ldots, L_n \in S$ such that $F_1 L_1^{d_1}, F_2 L_2^{d_2}, \ldots, F_n L_n^{d_n} \in I$ is again a regular sequence for any integers $d_1, \ldots, d_n \geq 0$.*

Proof. We prove the lemma in the case of three variables. The proof in general follows the same outline. Let $d_1, d_2, d_3 \geq 0$ be integers.

We know that $S/(F_1, F_2)$ has a non-zero-divisor. Hence, there exists a general linear form $L_3 \in S$ such that L_3 is a non-zero-divisor on $S/(F_1, F_2)$. Thus, $F_3 L_3^{d_3}$ is a non-zero-divisor on $S/(F_1, F_2)$, i.e. $F_1, F_2, F_3 L_3^{d_3}$ is a regular sequence of forms. The conclusion is now obtained by permuting the order of regular sequences and repeating this argument. □

Let R, S and I be as in Standing Notation 3.2 but replace the degrees of the generators of I with $\deg(F) = l \geq 4$, $\deg(G) = m \geq l$ and $\deg(H_i) = m + 1$ for all i. We fix $J \subseteq R$ to be the ideal generated by x^{m+1}, y^{m+1} and the $\dim_k(I_{m+1}) - 2$ *largest* monomials, with respect to the degree-lexicographic ordering, in $R_{m+1} \setminus \{x^{m+1}, y^{m+1}\}$.

Theorem 4.2. *If the last generator of J is among the last $m-3-j$ elements of Group $(m-2-j)$, for $0 \leq j \leq m-4$, then* $\dim_k(R_1 I_{m+1}) \geq \dim_k(R_1 J_{m+1}) + 1$.

The proof of Theorem 4.2 is similar to that of Theorem 3.3. We use Lemma 4.1 to guarantee an appropriate regular sequence in degree $t = m + 1$ and then apply Theorem 3.3 to obtain a first rough bound for $\dim_k(R_1 I_{m+1})$. Due to length, the arguments are omitted but can be found in [4], along with other such generalizations.

Acknowledgements

I gratefully acknowledge the financial support I received from both Queen's University and the Natural Sciences and Engineering Research Council of Canada. My sincerest thanks go to my Ph.D. supervisors Tony Geramita and Leslie Roberts for their guidance. I also wish to thank Mike Roth, Greg Smith and Hema Srinivasan for their many valuable comments.

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