1. Let $n$ be any positive integer. Show that $\{1, \cos(t), \cos^2(t), \cos^3(t), \ldots, \cos^n(t)\}$ is a linearly independent subset of $F$ (the vector space of all functions). Here is how you can do this: Learn about Vandermonde determinants. (These are discussed briefly on page 288 of your text, or you can learn about them from a variety of other sources.) Now consider a hypothetical dependence relation of the form

$$c_1 + c_2 \cos(t) + c_3 \cos^2(t) + \cdots + c_{n+1} \cos^n(t) = 0.$$  

Pick $n+1$ values of $t$, say $t = t_1, t_2, \ldots, t_{n+1}$, such that $\cos(t_1), \ldots, \cos(t_{n+1})$ are distinct numbers. (We’ll take it as obvious that such $t_i$’s exist.) This gives $n+1$ equations involving the constants $c_1, \ldots, c_{n+1}$. Now use what you learned about Vandermonde determinants to conclude that we must have $c_1 = c_2 = \cdots = c_{n+1} = 0$.

**Solution:** Suppose we have scalars $c_1, c_2, \ldots, c_{n+1}$ such that

$$c_1(1) + c_2 \cos(t) + c_3 \cos^2(t) + \cdots + c_{n+1} \cos^n(t) = 0,$$  

where 0 denotes the zero function $f$ defined by $f(t) = 0$ for all $t$.

As suggested, we pick $n+1$ values of $t$, say $t = t_1, t_2, \ldots, t_{n+1}$ such that $\cos(t_1), \cos(t_2), \ldots, \cos(t_{n+1})$ are distinct numbers. Plugging these values of $t$ into the above equation gives us the following $n+1$ equations in the variables $c_1, c_2, \ldots, c_{n+1}$:

$$0 = (1)c_1 + \cos(t_1)(c_2) + \cos^2(t_1)(c_3) + \cdots + \cos^n(t_1)(c_{n+1})$$
$$0 = (1)c_1 + \cos(t_2)(c_2) + \cos^2(t_2)(c_3) + \cdots + \cos^n(t_2)(c_{n+1})$$
$$\cdots$$
$$0 = (1)c_1 + \cos(t_{n+1})(c_2) + \cos^2(t_{n+1})(c_3) + \cdots + \cos^n(t_{n+1})(c_{n+1})$$

We can express this system of linear equations in the matrix equation $Ac = 0$ where

$$A = \begin{bmatrix}
1 & \cos(t_1) & \cos^2(t_1) & \cdots & \cos^n(t_1) \\
1 & \cos(t_2) & \cos^2(t_2) & \cdots & \cos^n(t_2) \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
1 & \cos(t_n) & \cos^2(t_n) & \cdots & \cos^n(t_n) \\
1 & \cos(t_{n+1}) & \cos^2(t_{n+1}) & \cdots & \cos^n(t_{n+1})
\end{bmatrix}$$

and

$$c = \begin{bmatrix}
c_1 \\
c_2 \\
\vdots \\
c_n \\
c_{n+1}
\end{bmatrix}.$$  

Using Vandermonde determinants, we have that

$$\det(A) = \prod_{1 \leq i < j \leq n+1} (\cos(t_j) - \cos(t_i))$$
$$= (\cos(t_2) - \cos(t_1)) \cdots (\cos(t_{n+1}) - \cos(t_1))(\cos(t_3) - \cos(t_2)) \cdots (\cos(t_{n+1}) - \cos(t_2)) \cdots (\cos(t_{n+1}) - \cos(t_n))$$
Since the numbers \( \cos(t_1), \cos(t_2), \ldots, \cos(t_{n+1}) \) are distinct, the number \( \cos(t_j) - \cos(t_i) \neq 0 \) for all \( 1 \leq i < j \leq n + 1 \). Thus, \( \det(A) \neq 0 \). We conclude that the \( (n + 1) \times (n + 1) \) matrix \( A \) is invertible. This means that the system \( Ac = 0 \) has the unique solution

\[
c = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \\ c_{n+1} \end{bmatrix} = A^{-1}0 = 0.
\]

Since \( c = 0 \), we must have that

\[
c_1 = c_2 = c_3 = \cdots = c_n = c_{n+1} = 0.
\]

Therefore,

\[
B = \{1, \cos(t), \cos^2(t), \ldots, \cos^n(t)\}
\]

is a linearly independent subset of \( \mathcal{F} \).

2. Let \( V \) be the subspace of \( \mathcal{F} \) spanned by \( B = \{1, \cos(t), \cos^2(t), \cos^3(t), \cos^4(t), \cos^5(t)\} \). Since \( B \) is linearly independent (as you showed in (1)), we have that \( B \) is a basis of \( V \). Using the trigonometric identities

\[
\begin{align*}
\cos(2t) &= -1 + 2 \cos^2(t) \\
\cos(3t) &= -3 \cos(t) + 4 \cos^3(t) \\
\cos(4t) &= 1 - 8 \cos^2(t) + 8 \cos^4(t) \\
\cos(5t) &= 5 \cos(t) - 20 \cos^3(t) + 16 \cos^5(t)
\end{align*}
\]

write the \( B \)-coordinate vector for each of the functions \( 1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t) \).

**Solution:** Using the trig identities, we express the functions \( 1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t) \) as linear combinations of the functions in \( B \):

\[
\begin{align*}
1 &= 1(1) + 0 \cos(t) + 0 \cos^2(t) + 0 \cos^3(t) + 0 \cos^4(t) + 0 \cos^5(t) \\
\cos(t) &= 0(1) + 1 \cos(t) + 0 \cos^2(t) + 0 \cos^3(t) + 0 \cos^4(t) + 0 \cos^5(t) \\
\cos(2t) &= -1(1) + 0 \cos(t) + 2 \cos^2(t) + 0 \cos^3(t) + 0 \cos^4(t) + 0 \cos^5(t) \\
\cos(3t) &= 0(1) - 3 \cos(t) + 0 \cos^2(t) + 4 \cos^3(t) + 0 \cos^4(t) + 0 \cos^5(t) \\
\cos(4t) &= 1(1) + 0 \cos(t) - 8 \cos^2(t) + 0 \cos^3(t) + 8 \cos^4(t) + 0 \cos^5(t) \\
\cos(5t) &= 0(1) + 5 \cos(t) + 0 \cos^2(t) - 20 \cos^3(t) + 0 \cos^4(t) + 16 \cos^5(t)
\end{align*}
\]

Thus, by definition, we have the following \( B \)-coordinate vectors:

\[
[1]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\cos(t)]_B = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\cos(2t)]_B = \begin{bmatrix} -1 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\cos(3t)]_B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 0 \end{bmatrix}, \quad [\cos(4t)]_B = \begin{bmatrix} 1 \\ 0 \\ -8 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad [\cos(5t)]_B = \begin{bmatrix} 0 \\ 0 \\ 8 \\ 4 \\ 0 \\ 0 \end{bmatrix}
\]

Since \( [1]_B \neq 0 \), the \( B \)-coordinate vectors are distinct, the number \( \cos(t_i) \) is linearly independent. Thus, the \( (n + 1) \times (n + 1) \) matrix \( A \) is invertible.
3. Use the calculations from the previous part to show that \( C = \{1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\} \) is another basis of \( V \).

Solution: Let \( A \) be the \( 6 \times 6 \) matrix whose columns are the coordinate vectors we found in Task 2. That is, let

\[
A = \begin{bmatrix}
1 & 0 & 1 & 0 & 1 & 0 \\
0 & 1 & 0 & 3 & 0 & 5 \\
0 & 0 & 2 & 0 & -8 & 0 \\
0 & 0 & 0 & 0 & 4 & 0 \\
0 & 0 & 0 & 0 & 8 & 0 \\
0 & 0 & 0 & 0 & 0 & 16
\end{bmatrix}.
\]

We see that \( A \) is already in row-echelon form. Moreover, since there is a pivot (i.e., a leading entry) in each column of \( A \), we know that the column vectors of \( A \) form a linearly independent set in \( \mathbb{R}^6 \). Therefore, by Theorem 6.7, the set \( C = \{1, \cos(t), \cos(2t), \cos(3t), \cos(4t), \cos(5t)\} \) is a linearly independent set of functions.

Knowing that \( B \) is a basis for \( V \), we see that \( \dim(V) = 6 \). Since the functions in \( C \) are linear combinations of the functions in \( B \), we know that the functions in \( C \) are also in \( V \). Therefore, since \( C \) has \( 6 = \dim(V) \) linearly independent functions in \( V \), \( C \) must also be a basis for \( V \) by Theorem 6.10 (c).

4. Use the calculations from (2) to find the change of basis matrix \( P_{B \leftarrow C} \) and then use a calculator to find \( P_{C \leftarrow B} \).

Solution: By definition,

\[
P_{B \leftarrow C} = \begin{bmatrix}
[1]_B & [\cos(t)]_B & [\cos(2t)]_B & [\cos(3t)]_B & [\cos(4t)]_B & [\cos(5t)]_B \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

By Theorem 6.12 (c), \( P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1} \). Using a CAS we find

\[
P_{C \leftarrow B} = (P_{B \leftarrow C})^{-1} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}.
\]

5. Use \( P_{C \leftarrow B} \) to calculate

\[
\int \left( a_0 + a_1 \cos(t) + a_2 \cos^2(t) + a_3 \cos^3(t) + a_4 \cos^4(t) + a_5 \cos^5(t) \right) \, dt
\]

where \( a_0, \ldots, a_5 \) are arbitrary constants, by first transforming the integrand into a linear combination of the functions in \( C \).

Solution: Let \( f(t) = a_0 + a_1 \cos(t) + a_2 \cos^2(t) + a_3 \cos^3(t) + a_4 \cos^4(t) + a_5 \cos^5(t) \). Then

\[
[f(t)]_B = \begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}.
\]
By Theorem 6.12 (a),

\[
[f(t)]_C = P_{C \rightarrow B}[f(t)]_B
\]

\[
= \begin{bmatrix}
1 & 0 & \frac{1}{2} & 0 & \frac{3}{8} & 0 \\
0 & 1 & 0 & \frac{3}{4} & 0 & \frac{5}{8} \\
0 & 0 & 1 & \frac{1}{2} & 0 & 0 \\
0 & 0 & 0 & 1 & \frac{1}{4} & 0 \\
0 & 0 & 0 & 0 & 1 & \frac{5}{16}
\end{bmatrix}
\begin{bmatrix}
a_0 \\
a_1 \\
a_2 \\
a_3 \\
a_4 \\
a_5
\end{bmatrix}
\]

This means that

\[
f(t) = \left( a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4 \right) t + \left( a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5 \right) \cos(t) + \left( \frac{1}{2}a_2 + \frac{1}{2}a_4 \right) \cos(2t) \\
+ \left( \frac{1}{4}a_3 + \frac{5}{16}a_5 \right) \cos(3t) + \left( \frac{1}{8}a_4 \right) \cos(4t) + \left( \frac{1}{16}a_5 \right) \cos(5t).
\]

Therefore,

\[
\int f(t) \, dt = \left( a_0 + \frac{1}{2}a_2 + \frac{3}{8}a_4 \right) t + \left( a_1 + \frac{3}{4}a_3 + \frac{5}{8}a_5 \right) \sin(t) + \left( \frac{1}{2}a_2 + \frac{1}{2}a_4 \right) \frac{1}{2} \sin(2t) \\
+ \left( \frac{1}{4}a_3 + \frac{5}{16}a_5 \right) \frac{1}{3} \sin(3t) + \left( \frac{1}{8}a_4 \right) \frac{1}{4} \sin(4t) + \left( \frac{1}{16}a_5 \right) \frac{1}{5} \sin(5t) + C,
\]

where \( C \) is the constant of integration.