Taylor Series

1. A power series in powers of $x - c$ is an expression of the form

$$\sum_{k=0}^{\infty} b_k (x-c)^k. \tag{1}$$

The numbers b_k are the *coefficients* of the series. The set of x for which the series converges is called the *interval of convergence.*

2. **Proposition:** The power series (1) either

a. converges at $x = c$ and diverges everywhere else,

b. converges absolutely for all x, or

c. converges absolutely for $|x - c| < r$ and diverges for $|x - c| > r$, where $0 < r < \infty$. The endpoints $x = c \pm r$ must be tested separately for convergence.

The number r in part (c) is called the *radius of convergence.* In case (a) the radius of convergence is zero, and in case (b), infinity.

3. Within the interval of convergence the power series represents a function. Suppose that

$$f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k \tag{2}$$

has a positive radius of convergence. Within the interval of convergence you can differentiate and integrate the power series as you would a polynomial, term-by-term. Thus,

$$f'(x) = \sum_{k=1}^{\infty} k b_k (x-c)^{k-1}, \tag{3}$$

and

$$\int f(x)\, dx = \sum_{k=0}^{\infty} \frac{b_k}{k+1} (x-c)^{k+1} + K, \tag{4}$$

where K is the constant of integration. Term-by-term differentiation and integration do not change the radius of convergence, but might change the endpoint behavior. So, for example,

$$\frac{1}{1+x} = \sum_{k=0}^{\infty} (-1)^k x^k, \tag{5}$$

has interval of convergence $(-1, 1)$, but its integral

$$\ln(1+x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k+1} x^{k+1}, \tag{6}$$

converges on $(-1, 1]$.

4. You can manufacture other power series from the geometric series

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k \quad \text{for } |x| < 1.$$

Replace x with $-x$ to get the series (3). Integrate (3) to get the (4). Replace x with x^2 in (4) to get

$$\frac{1}{1+x^2} = \sum_{k=0}^{\infty} (-1)^k x^{2k} \quad \text{for } |x| < 1. \tag{7}$$

Integrate this to get

$$\arctan x = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1} x^{2k+1}, \quad \text{for } -1 < x \leq 1. \tag{8}$$

5. Let

$$f(x) = \sum_{k=0}^{\infty} b_k (x-c)^k, \tag{9}$$

have a nonzero radius of convergence. By differentiating k times and setting $x = c$, we find that

$$b_k = \frac{f^{(k)}(c)}{k!}. \tag{10}$$

So for x in the interval of convergence of the series (9),

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!} (x-c)^k. \tag{11}$$

When the power series is written in the form (11), it is called the *Taylor series for f about the point c.* Thus you could refer to the series on the right-hand side of (8) as the "power series" or the "Taylor series" for $\arctan x$ about c. The numbers b_k as given by (10) are called *Taylor coefficients.*

6. With the recipe (11) you can in principle compute any Taylor series. Some important examples are

a. The Taylor series for $\sin x$ about $c = 0$:

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \quad \text{for all } x. \tag{12}$$

b. The Taylor series for $\cos x$ about $c = 0$:

$$\cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!}\, x^{2k}, \quad \text{for all } x. \tag{13}$$

c. The Taylor series for e^x about $c = 0$:

$$e^x = \sum_{k=0}^{\infty} \frac{1}{k!}\, x^k, \quad \text{for all } x. \tag{14}$$

d. The Taylor series for $\cosh x$ about $c = 0$:

$$\cosh x = \sum_{k=0}^{\infty} \frac{1}{(2k)!}\, x^{2k}, \quad \text{for all } x. \tag{15}$$

e. The Taylor series for $\sinh x$ about $c = 0$:

$$\sinh x = \sum_{k=0}^{\infty} \frac{1}{(2k+1)!}\, x^{2k+1}, \quad \text{for all } x. \tag{16}$$

7. You can use the Taylor series (12)-(16) to manufacture other Taylor series. For example, by replacing x with $-x^2$ in (14), we get

$$e^{-x^2} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} x^{2k}, \quad \text{for all } x. \tag{17}$$

Multiply the series (13) by x^3 to obtain

$$x^3 \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k+3}, \quad \text{for all } x. \tag{18}$$

In both these examples, it is easier to build the new series from the old than it is to use the recipe (11).

8. By dropping terms of order larger than n, we obtain the nth *Taylor polynomial* for f about c.

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(c)}{k!} (x-c)^k. \tag{19}$$

Note that $P_0(x)$ is the constant $f(c)$ and that

$$P_1(x) = f(c) + f'(c)(x-c), \tag{20}$$

is the tangent line approximation to f at $x = c$. In general, $P_n(x)$ is the polynomial of degree at most n that best approximates $f(x)$ for x *near* c.

9. What is meant by "best approximation," in the preceeding paragraph? By differentiating $P_n(x)$ k times and setting $x = c$ we see that

$$P_n^{(k)}(c) \equiv f^{(k)}(c), \quad \text{for } k = 0, \ldots, n \tag{21}$$

Thus the values of P_n and f, and those of their derivatives up to order n, coincide at $x = c$. Thus for n large, once expects the graph of P_n to resemble that of f when x is close to c. Since $P_n(x)$ is a polynomial of degree n,

$$P_n^{(m)}(x) \equiv 0,$$

for $m > n$. Thus you can't count on matching the derivatives of P_n and f at c beyond order n.

10. By increasing n you can improve the approximation

$$P_n(x) \approx f(x), \tag{22}$$

and enlarge the region over which it is valid. However, this will only work inside the interval of convergence. For x outside the interval, $P_n(x)$ won't generally be a good approximation to $f(x)$, and you won't be able to improve the situation by increasing n.

11. How good is the approximation (22)? Taylor's theorem asserts that

$$\begin{aligned} f(x) - P_n(x) &= R_n(x) \\ &= \frac{f^{(n+1)}(z)}{(n+1)!}\,(x-c)^{n+1}, \end{aligned} \tag{23}$$

where z lies between c and x. $R_n(x)$ is called the *remainder*.