Basic Matrix Operations

1. Let $A = (a_{ij})$ be an $m \times n$ matrix:

$$A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}.$$

2. The *transpose* of A is the $n \times m$ matrix

$$A^t = \begin{bmatrix} a_{11} & \cdots & a_{m1} \\ \vdots & \ddots & \vdots \\ a_{1n} & \cdots & a_{mn} \end{bmatrix}.$$

The rows of A are the columns of A^t. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \tag{1}$$

then

$$A^t = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}.$$

Note that $(A^t)^t = A$.

3. If c is a scalar (i.e. a real or complex number) and $A = (a_{ij})$is an $m \times n$ matrix, then cA is the $m \times n$ matrix with entries ca_{ij}. So if A is given by (1) and $c = 2$, then

$$cA = 2\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}.$$

4. If $A = (a_{ij})$ and $B = (b_{ij})$ are $m \times n$ matrices, then the sum $A + B$ is the $m \times n$ matrix with entries $a_{ij} + b_{ij}$. So if A is given by (1) and

$$B = \begin{bmatrix} -1 & 3 & -2 \\ 0 & 1 & -4 \end{bmatrix}, \tag{2}$$

then

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} -1 & 3 & -2 \\ 0 & 1 & -4 \end{bmatrix} = \begin{bmatrix} 0 & 5 & 1 \\ 4 & 6 & 2 \end{bmatrix}.$$

5. If A is an $m \times n$ matrix and B is $n \times k$, then the product AB is the $m \times k$ matrix whose ijth entry is the dot product of the ith row of A and the jth column of B. For example, let

$$A = \begin{bmatrix} 1 & 1 & 4 \\ 2 & 5 & -1 \end{bmatrix} \tag{3}$$

$$B = \begin{bmatrix} 2 & -2 & 1 \\ 0 & 3 & 1 \\ -3 & 1 & 2 \end{bmatrix}. \tag{4}$$

Since A is 2×3 and B 3×3, the product AB will be a 2×3 matrix. In particular,

$$AB = \begin{bmatrix} -10 & 5 & 10 \\ 7 & 10 & 5 \end{bmatrix}.$$

6. The $m \times m$ *identity* matrix I_m is the $m \times m$ matrix with main diagonal entries 1 and 0's everywhere else. So, for example,

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \quad (5)$$

7. An $m \times m$ matrix A is *invertible*, or *nonsingular* if there is an $m \times m$ matrix A^{-1} such that

$$AA^{-1} = A^{-1}A = I_m.$$

The matrix A^{-1} is called the *inverse* of A.

8. The determinant of the 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

is

$$\det(A) = ab - cd.$$

If $\det(A) \neq 0$, then A is invertible, and its inverse is

$$\frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Since the matrix

$$A = \begin{bmatrix} 4 & 2 \\ 1 & 2 \end{bmatrix},$$

has $\det(A) = 6$, it is invertible and

$$A^{-1} = \frac{1}{6} \begin{bmatrix} 2 & -2 \\ -1 & 4 \end{bmatrix}.$$