Second-Order, Linear Equations 3: The Inhomogeneous Equation

1. Consider the linear, second-order, inhomogeneous equation

$$Lu \equiv u'' + p(t)u' + q(t)u = g(t), \tag{1}$$

for t in some open interval I. The homogeneous equation is

$$Lu \equiv u'' + p(t)u' + q(t)u = 0. \tag{2}$$

2. **Proposition:** Let $\{u_1, u_2\}$ be a fundamental set for the homogeneous equation (2), and u_p a solution to the inhomogeneous equation (1). Then the general solution to (1) is

$$u(t) = c_1 u_1(t) + c_2 u_2(t) + u_p(t). \tag{3}$$

3. You can find a particular solution u_p to the inhomogeneous equation by the method of *variation of parameters.*

a. Let u_1 and u_2 be independent solutions to (2). Look for the particular solution in the form

$$u_p(t) = c_1(t)u_1(t) + c_2(t)u_2(t). \tag{4}$$

b. If you *assume* that

$$c_1' u_1 + c_2' u_2 = 0, \tag{5}$$

then the equation

$$Lu_p = g(t),$$

reduces to

$$c_1' u_1' + c_2' u_2' = g(t). \tag{6}$$

You now have two equations, (5) and (6), for the two unknowns c_1' and c_2'. The solutions are

$$c_1'(t) = -\frac{g(t)u_2(t)}{W(u_1, u_2)(t)}, \tag{7}$$

$$c_2'(t) = \frac{g(t)u_1(t)}{W(u_1, u_2)(t)}. \tag{8}$$

Since u_1 and u_2 are linearly independent, the Wronskian in the denominator is nonzero. Thus

$$c_1(t) = -\int \frac{g(t)u_2(t)}{W(u_1, u_2)(t)}\, dt, \tag{9}$$

and

$$c_2(t) = \int \frac{g(t)u_1(t)}{W(u_1, u_2)(t)}\, dt. \tag{10}$$

You can take the constants of integration in (9) and (10) to be zero. When you can't do the integrals, it is best to write $c_1(t)$ annd $c_2(t)$ as definite integrals.

c. Form the general solution,

$$\begin{aligned} u(t) &= a_1 u_1(t) + a_2 u_2(t) + u_p(t) \\ &= a_1 u_1(t) + a_2 u_2(t) + c_1(t) u_1(t) + c_2(t) u_2(t), \end{aligned} \tag{11}$$

where $c_1(t)$ and $c_2(t)$ are given by (9) and (10).

4. **Example:** Find the general solution to

$$u'' + u = \sin 2t. \tag{12}$$

Two independent solutions to the homogeneous equation $u'' + u = 0$ are $u_1(t) = \cos t$ and $u_2(t) = \sin t$. The Wronskian is

$$W(u_1, u_2)(t) = 1. \tag{13}$$

Hence

$$\begin{aligned} c_1(t) &= -\int \frac{g(t) u_2(t)}{W(u_1, u_2)(t)}\, dt \\ &= -\int \sin 2t \cos t \, dt \\ &= \frac{2}{3} \sin^3 t, \end{aligned} \tag{14}$$

and

$$\begin{aligned} c_2(t) &= \int \frac{g(t) u_1(t)}{W(u_1, u_2)(t)}\, dt \\ &= \int \sin 2t \sin t \, dt \\ &= \frac{2}{3} \sin^3 t, \end{aligned} \tag{15}$$

Hence the general solution to (12) is

$$\begin{aligned} u(t) &= a_1 \cos t + a_2 \sin t + \frac{2}{3} \sin^3 t \cos t + \frac{2}{3} \cos^3 t \sin t \\ &= a_1 \cos t + a_2 \sin t + \frac{1}{3} \sin 2t. \end{aligned} \tag{16}$$

5. **Example:** Solve the initial value problem

$$\begin{cases} y'' - y = \exp(\sin t), \\ y(0) = 1, \\ y'(0) = 0. \end{cases}$$

Two independent solutions to the homogeneous equation are $y_1(t) = e^{-t}$ and $y_2(t) = e^t$. The Wronskian is

$$W(y_1, y_2)(t) = 2.$$

Hence

$$c_1(t) = -\frac{1}{2}\int_0^t e^{z+\sin z}\,dz,$$

and

$$c_2(t) = \frac{1}{2}\int_0^t e^{-z+\sin z}\,dz.$$

The general solution is

$$y(t) = a_1 e^{-t} + a_2 e^t - \frac{e^{-t}}{2}\int_0^t e^{z+\sin z}\,dz + \frac{e^t}{2}\int_0^t e^{-z+\sin z}\,dz. \tag{17}$$

The initial conditions tell you that

$$y(0) = a_1 + a_2 = 1, \tag{18}$$

and

$$y'(0) = -a_1 + a_2 = 0, \tag{19}$$

and hence that

$$a_1 = a_2 = \frac{1}{2}.$$

$$y(t) = \cosh t - \frac{e^{-t}}{2}\int_0^t e^{z+\sin z}\,dz + \frac{e^t}{2}\int_0^t e^{-z+\sin z}\,dz. \tag{20}$$

This can also be written as

$$y(t) = \cosh t + \int_0^t \sinh(t-z)\, e^{\sin z}\,dz. \tag{21}$$