Second-Order, Linear Equations 2

1. We consider the problem of assembling a fundamental set $\{u_1(t), u_2(t)\}$ for the second-order, linear, homogeneous equation

$$Lu = u'' + p(t)u' + q(t)u = 0. \tag{1}$$

With the existence-uniqueness theorem, you can *prove* the existence of a fundamental set. Let $u_1(t)$ be the solution to the initial value problem

$$\begin{cases} Lu = 0, & \text{for } t \in I, \\ u(\tau) = 1, \\ u'(\tau) = 0, \end{cases} \tag{2}$$

and $u_2(t)$ to

$$\begin{cases} Lu = 0, & \text{for } t \in I, \\ u(\tau) = 0, \\ u'(\tau) = 1. \end{cases} \tag{3}$$

Then u_1 and u_2 both satisfy the linear, homogeneous equation. The Wronskian at τ is

$$W(u_1, u_2)(\tau) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

which implies the linear independence of u_1 and u_2 on I. Thus $\{u_1, u_2\}$ is a fundamental set for equation (1) on the interval I. *Finding* u_1 and u_2 analytically is another matter altogether. It is usually impossible. In these notes we'll consider those few cases in which it can be done.

2. Reduction of Order: If you already have a nontrivial solution u_1, you can use *reduction of order* to find a second one, u_2, that is linearly independent of the first.

Look for the second solution in the form

$$u_2(t) = v(t)u_1(t). \tag{4}$$

Plug u_2 into equation (1), and use the fact that $Lu_1 = 0$ to get

$$v'' + \left(\frac{2u_1'}{u_1} + p\right) v' = 0. \tag{5}$$

Reduce the order of (5) by setting

$$w = v'. \tag{6}$$

Thus w satisfies the first-order, linear, homogeneous equation

$$w' + \left(\frac{2u_1'}{u_1} + p\right) w = 0. \tag{7}$$

Let $P(t)$ be an antiderivative of $p(t)$. The general solution to (7) is

$$w(t) = \frac{Ce^{-P(t)}}{u_1(t)^2}. \tag{8}$$

And by (4),

$$v(t) = C \int \frac{e^{-P(t)}}{u_1(t)^2}\, dt. \tag{9}$$

I've taken the constant of integration in (9) to be zero. It is sometimes helpful, especially when you can't do the integral, to write it in the definite form,

$$v(t) = C \int_{\tau}^{t} \frac{e^{-P(s)}}{u_1(s)^2}\, ds, \tag{10}$$

where τ is some fixed value of t. Plug (7) or (8) into (2), and you have your second solution u_2. To see that u_1 and u_2 are linearly independent, just note that $v(t)$ is not a constant. So by (2), neither u_1 nor u_2 is a constant multiple of the the other.

3. In the constant-coefficients case,

$$Lu = u'' + pu' + qu = 0, \tag{11}$$

you can always find two linearly indepedent solutions. Look for solutions in the form $u = e^{\lambda t}$. Plug this into equation (11) to get

$$(\lambda^2 + p\lambda + q)\, e^{\lambda t} = 0.$$

Thus $u = e^{\lambda t}$ satisfies (9) if and only if

$$\lambda^2 + p\lambda + q = 0. \tag{12}$$

The is called the characteristic equation.

a. When $p^2 - 4q > 0$, the characteristic equation (10) has real, distinct roots λ_1 and λ_2. Thus

$$u_1(t) = e^{\lambda_1 t}, \tag{13}$$

and

$$u_2(t) = e^{\lambda_2 t}, \tag{14}$$

are solutions to (9). It is easily checked (by inspection or with the Wronskian) that they are linearly independent. Thus the general solution to (9) is

$$u(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}. \tag{15}$$

b. When $p^2 - 4q = 0$, the characteristic equation has one root: $\lambda = -\frac{p}{2}$. Thus

$$u_1(t) = e^{-\frac{p}{2}t}, \tag{16}$$

satisfies (9). Reduction of order yields a second solution

$$u_2(t) = te^{-\frac{p}{2}t},$$

that is independent of the first. Thus the general solution to (9) is

$$u(t) = c_1 e^{-\frac{p}{2}t} + c_2 t e^{-\frac{p}{2}t}. \tag{17}$$

c. When $p^2 - 4q < 0$, the characteristic equation has roots $\lambda = \alpha + i\beta$ and $\bar{\lambda} = \alpha - i\beta$. Thus

$$v_1(t) = e^{\lambda t} = e^{\alpha t} e^{i\beta t},$$

and

$$v_2(t) = e^{\bar{\lambda} t} = e^{\alpha t} e^{-i\beta t},$$

are independent solutions to (9). By the principle of superposition,

$$u_1(t) = \frac{v_1(t) + v_2(t)}{2} = e^{\alpha t} \cos \beta t, \tag{18}$$

and

$$u_2(t) = \frac{v_1(t) - v_2(t)}{2i} = e^{\alpha t} \sin \beta t, \tag{19}$$

are also solutions. It is easily checked that they are linearly independent. Thus the general solution to (9) is

$$u(t) = c_1 e^{\alpha t} \cos \beta t + c_2 e^{\alpha t} \sin \beta t. \tag{20}$$

This solution is sometimes written in another form. Set

$$A = \sqrt{c_1^2 + c_2^2}.$$

Note that

$$\left(\frac{c_1}{A}\right)^2 + \left(\frac{c_2}{A}\right)^2 = 1.$$

We can thus set

$$\frac{c_1}{A} = \cos \varphi \quad \text{and} \quad \frac{c_2}{A} = \sin \varphi.$$

for some angle φ. Hence

$$\begin{aligned} u(t) &= A e^{\alpha t} \left[\cos \varphi \, \cos \beta t + \sin \varphi \, \sin \beta t\right] \\ &= A e^{\alpha t} \cos\left(\beta t - \varphi\right). \end{aligned} \tag{21}$$

4. Power series solutions: When your linear equation has the form

$$p_2(t) u'' + p_1(t) u' + p_0(t) u = 0, \tag{22}$$

where p_2, p_1 and p_0 are polynomials you can sometimes find solutions in the form of power series. Suppose that you want to solve (22) on some open interval I. Choose τ in I, and assume that solutions u_1 and u_2 have power series representations

$$u_1(t) = a_0 + a_1(t - \tau) + a_2(t - \tau)^2 + \cdots \tag{23}$$

and

$$u_2(t) = b_0 + b_1(t - \tau) + b_2(t - \tau)^2 + \cdots . \tag{24}$$

To insure that these solutions be independent, take

$$u_1(\tau) = a_0 = 1 \quad \text{and} \quad u_1'(\tau) = a_1 = 1, \tag{25}$$

and

$$u_2(\tau) = b_0 = 0 \quad \text{and} \quad u_2'(\tau) = b_1 = 0. \tag{26}$$

Then the Wronskian at τ is

$$W(u_1, u_2)(\tau) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1,$$

which implies the linear independence of u_1 and u_2 on I. You start by plugging the power series (23) into equation (22). Try to write the resulting equation as a single series set equal to zero. From this you should be able to extract a recurrence relation for the coefficients a_j. Then use the recurrence relation and (23) solve for the a_j. Use the same recipe to determine the b_j. A few points:

a. The method of series solutions is only practical in a few cases where the polynomials p_2, p_1 and p_0 are *very* simple. As we saw in class, the method worked on the Airy equation

$$u'' - tu = 0, \tag{27}$$

where $p_2(t) = 1$, $p_1(t) = 0$ and $p_0(t) = t$.

b. When solving an initial value problem, you should choose the point τ in (21) and (22) to be one at which the initial values are given.

c. The power series might only converge in a small interval centered at τ.

5. A Cauchy-Euler (equidimensional) equation is one of the form

$$at^2u'' + btu' + cu = 0. \tag{28}$$

For an equation of this sort, you can always find a fundamtental set. Look for solutions of the form $u = t^m$. Plug this into (28) to obtain

$$[am^2 + (b-a)m + c]t^m = 0. \tag{29}$$

If we take $t > 0$, then (29) implies that

$$am^2 + (b-a)m + c = 0. \tag{30}$$

Thus $u = t^m$ satisfies (26) if and only if m is a root of the characteristic equation (28).

a. If (28) has real, distinct roots m_1 and m_2, then for $t > 0$,

$$u_1(t) = t^{m_1}, \tag{31}$$

and

$$u_2(t) = t^{m_2}, \tag{32}$$

satisfy (26). It is easily seen that, since $m_1 \neq m_2$, these two solutions are independent. Hence the general solution to (26) is

$$u(t) = c_1 t^{m_1} + c_2 t^{m_2}, \tag{33}$$

for $t > 0$.

b. If (28) has one root m, then $u_1(t) = t^m$ satisfies (26). Reduction of order yields a second solution,

$$u_2(t) = t^m \ln t,$$

that is linearly independent of $u_1(t)$. Thus, for $t > 0$, the general solution to (26) in this case is

$$u(t) = c_1 t^m + c_2 t^m \ln t. \tag{34}$$

c. If (28) has complex roots $m = \alpha + i\beta$ and $\bar{m} = \alpha - i\beta$, then

$$v_1(t) = t^m = t^\alpha t^{i\beta} = t^\alpha e^{i\beta \ln t},$$

and

$$v_2(t) = t^{\bar{m}} = t^\alpha t^{-i\beta} = t^\alpha e^{-i\beta \ln t},$$

are linearly independent solutions to (26) for $t > 0$. By the superposition principle,

$$u_1(t) = \frac{v_1(t) + v_2(t)}{2} = t^\alpha \cos(\beta \ln t), \tag{35}$$

and

$$u_1(t) = \frac{v_1(t) - v_2(t)}{2} = t^\alpha \sin(\beta \ln t), \tag{36}$$

are also satisfy (26). It is easily checked that they are linearly independent. Thus, for $t > 0$, the general solution to (26) is

$$u(t) = c_1 t^\alpha \cos(\beta \ln t) + c_2 t^\alpha \sin(\beta \ln t). \tag{37}$$