Some Laplace Transform Practice Problems

1. Let $\mathcal{L}\{f(t)\} = F(s)$ and $\mathcal{L}\{g(t)\} = G(s)$. Prove that

$$\mathcal{L}\{(f * g)(t)\} = F(s)G(s).$$

2. Let $\mathcal{L}\{u(t)\} = U(s)$. Prove that

$$\mathcal{L}\{tu(t)\} = -U'(s).$$

3. Let $\mathcal{L}\{x(t)\} = X(s)$. Prove that

$$\mathcal{L}\left\{e^{at}x(t)\right\} = X(s-a).$$

4. Let $\mathcal{L}\{f(t)\} = F(s)$. Prove that

$$\mathcal{L}\{h_a(t)f(t-a)\} = e^{-as}F(s).$$

5. Let $\mathcal{L}\{z(t)\} = Z(s)$. Prove that

$$\mathcal{L}\{z''(t)\} = -z'(0) - sz(0) + s^2 Z(s).$$

6. Solve the initial value problem

$$\begin{cases} u'' - 3u' + 2u = 1 + t, \\ u(0) = 0, \quad u'(0) = 1. \end{cases}$$

7. Solve the initial value problem

$$\begin{cases} u'' + 4u = e^{-t}, \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

8. Solve the initial value problem

$$\begin{cases} u'' + 4u = [1 - h_{2\pi}(t)] \cos 2t, \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

9. Let $\mathcal{L}\{u(t)\} = U(s)$. Prove that

$$\mathcal{L}\left\{\frac{u(t)}{t}\right\} = \int_s^\infty U(s)\, ds.$$

10. Show that

$$\mathcal{L}\left\{t^{-\frac{1}{2}}\right\} = \sqrt{\frac{\pi}{s}}.$$

You may use Liouville's theorem:

$$\int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{2}.$$

11. The Dirac delta function $\delta_a(t)$ is defined by the properties

$$\delta_a(t) = \begin{cases} 0 & \text{for } t \neq a, \\ \infty & \text{for } t = a, \end{cases} \tag{1}$$

and

$$\int_0^\infty \delta_a(t)\, dt = 1. \tag{2}$$

a. Give a *clear* physical justification for defining such a function. In particular, why should we stipulate (2) if $\delta_a(t) = 0$ except at $t = a$?

b. Use (1) and (2) to show that

$$\int_0^\infty \delta_a(t) f(t)\, dt = f(a).$$

12. Solve the initial value problem

$$\begin{cases} u'' - 4u = f(t), \\ u(0) = 0, \quad u'(0), \end{cases}$$

where

$$f(t) = \begin{cases} t & \text{for } 0 \leq t < 1, \\ 2 & \text{for } 1 \leq t. \end{cases}$$

13. Solve the initial value problem

$$\begin{cases} u'' + 4u = \sin t + \delta_\pi(t), \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$

14. Solve the initial value problem

$$\begin{cases} x'' + 3x' + 2x = \delta_1(t), \\ u(0) = 0, \quad u'(0) = 0. \end{cases}$$