The Laplace Transform

1. The Laplace transform of a function \( f(t) \) is

\[
\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) \, dt, \tag{1}
\]

defined for those values of \( s \) at which the integral converges. For example, the Laplace transform of \( f(t) = e^{at} \) is

\[
\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at} \, dt = \int_0^\infty e^{-(s-a)t} \, dt = (s-a)^{-1}, \quad \text{for } s > a. \tag{2}
\]

2. Note that the Laplace transform of \( f(t) \) is a function of \( s \). Hence the transform is sometimes denoted \( \mathcal{L}\{f(t)\}(s) \), \( \mathcal{L}\{f\}(s) \), or simply \( F(s) \).

3. Example: The Laplace transform of \( f(t) = 1 \) is

\[
F(s) = \int_0^\infty e^{-st} \, dt = s^{-1}, \quad \text{for } s > 0. \tag{3}
\]

You can integrate by parts obtain the Laplace transform of \( f(t) = t \):

\[
\mathcal{L}\{t\} = \int_0^\infty te^{-st} \, dt = s^{-2}, \quad \text{for } s > 0. \tag{4}
\]

Integrate by parts \( n \) times to get

\[
\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st} \, dt = \frac{n!}{s^{n+1}}, \quad \text{for } s > 0, \text{ and } n = 0, 1, 2, \ldots \tag{5}
\]

4. The Gamma function is

\[
\Gamma(x) = \int_0^\infty z^{x-1} e^{-z} \, dz, \quad \text{for } x > 0. \tag{6}
\]
We showed in class that
\[ \Gamma(x + 1) = x\Gamma(x). \] (7)
Thus \( \Gamma(x) \) is the continuous extension of the factorial. We also showed that for \( a > -1 \),
\[ \mathcal{L}\{\Gamma(t)\} = \frac{\Gamma(a + 1)}{s^{a+1}}, \quad \text{for } s > 0. \] (8)
As \( \Gamma(x) \) generalizes the factorial, the Laplace transform (8) generalizes (5).

5. Example: The Laplace transforms of \( \sin \beta t \) and \( \cos \beta t \) are
\[ \mathcal{L}\{\sin \beta t\} = \int_0^\infty e^{-st} \sin \beta t \, dt \]
\[ = \frac{\beta}{s^2 + \beta^2}, \] (9)
and
\[ \mathcal{L}\{\cos \beta t\} = \int_0^\infty e^{-st} \cos \beta t \, dt \]
\[ = \frac{s}{s^2 + \beta^2}, \] (10)
both for \( s > 0 \).

6. Proposition: The Laplace transform is a linear operator, that is, for functions \( f(t) \) and \( g(t) \) and any constant \( c \),
\[ \mathcal{L}\{f(t) + g(t)\} = \mathcal{L}\{f(t)\} + \mathcal{L}\{g(t)\}, \] (11)
and
\[ \mathcal{L}\{cf(t)\} = c\mathcal{L}\{f(t)\}. \] (12)

7. Example: From (11), (12) and (2) we get
\[ \mathcal{L}\{\sinh \beta t\} = \frac{\beta}{s^2 - \beta^2}, \] (13)
and
\[ \mathcal{L}\{\cosh \beta t\} = \frac{s}{s^2 - \beta^2}, \] (14)
both for \( s > |\beta| \).
8. **Example:** The Laplace transform of \( x(t) = 2t - e^{-3t} + 4 \cos \pi t \) is

\[
X(s) = \mathcal{L}\{2t - e^{-3t} + 4 \cos \pi t\} = 2\mathcal{L}\{t\} - \mathcal{L}\{e^{-3t}\} + 4\mathcal{L}\{\cos \pi t\} = \frac{2}{s^2} - \frac{1}{s + 3} + \frac{4s}{s^2 + \pi^2}. \tag{15}
\]

9. When applying the Laplace transform to a function \( f(t) \), we will assume that \( f \) is of \textit{exponential order} over \([0, \infty)\). This means that for some \( t_0 \), and constants \( M \) and \( \alpha \),

\[
|f(t)| \leq Me^{\alpha t}, \tag{16}
\]

for all \( t \geq t_0 \). In class we proved the following result.

10. **Proposition:** If \( f(t) \) satisfies (16) then its Laplace transform \( F(s) \) exists for \( s > \alpha \) and

\[
\lim_{s \to \infty} F(s) = 0. \tag{17}
\]

11. **Note:** Unless otherwise stated, we’ll assume any function to which we apply the Laplace transform to be of exponential order. We conclude this set of notes with a few important properties of the Laplace transform. The derivations, done in class, are quite simple.

12. The shift property: Let \( \mathcal{L}\{f(t)\} = F(s) \). Then

\[
\mathcal{L}\{e^{at}f(t)\} = F(s - a). \tag{18}
\]

So, for example,

\[
\mathcal{L}\{e^{-t}\cos 2t\} = \frac{s + 1}{(s + 1)^2 + 4}.
\]

13. The switching property: Let \( H(t) \) be the Heaviside function:

\[
H(t) = \begin{cases} 
0 & \text{for } t < 0, \\
1 & \text{for } t \geq 0,
\end{cases}
\]

and \( F(s) \) be the Laplace transform of \( f(t) \). Then

\[
\mathcal{L}\{H(t - a)f(t - a)\} = e^{-as}F(s). \tag{19}
\]
Thus,
\[ \mathcal{L}\{H(t-3)(t-3)^5\} = \frac{5! e^{-3s}}{s^6}. \]
And,
\[
\mathcal{L}\{H(t-3)^2\} = \mathcal{L}\{H(t-3)(t-3+3)^2\} \\
= \mathcal{L}\{H(t-3)[(t-3)^2 + 6(t-3) + 9]\} \\
= e^{-3s} \left[ \frac{2}{s^4} + \frac{6}{s^2} + \frac{9}{s} \right]. \tag{20}
\]
Note: Some authors write \(h_a(t)\) or \(u_a(t)\) instead of \(H(t-a)\).

14. If \(F(s)\) is the Laplace transform of \(f(t)\), and \(n\) is a nonnegative integer, then
\[ \mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s). \tag{21} \]
Thus,
\[ \mathcal{L}\{t \sin 2t\} = \frac{4s}{(s^2 + 4)^2}. \]

15. Let \(F(s) = \mathcal{L}\{f(t)\}\). Then
\[ \mathcal{L}\{f'(t)\} = -f(0) + sF(s), \tag{22} \]
and
\[ \mathcal{L}\{f''(t)\} = -sf(0) - f'(0) + s^2 F(s). \tag{23} \]

16. The convolution of functions \(f(t)\) and \(g(t)\) is
\[ (f \ast g)(t) = \int_0^t f(t-u)g(u) \, du. \tag{24} \]
As we showed in class, convolution is commutative, i.e.
\[ (f \ast g)(t) = \int_0^t f(t-u)g(u) \, du = \int_0^t f(u)g(t-u) \, du = (g \ast f)(t). \tag{25} \]

17. Proposition: If \(F(s)\) and \(G(s)\) are the Laplace transforms of \(f(t)\) and \(g(t)\) respectively, then
\[ \mathcal{L}\{(f \ast g)(t)\} = F(s)G(s). \tag{26} \]

18. We can use the above proposition to compute the Laplace transform of \(\int_0^t f(u) \, du\). We can write this integral as \((f \ast 1)(t)\) and then apply (26):
\[ \mathcal{L}\left\{\int_0^t f(u) \, du\right\} = \mathcal{L}\{(f \ast 1)(t)\} = \frac{F(s)}{s}. \tag{27} \]