Some "Tricks" for Integration

Trick | **Examples**

Expand

$$\int (1+e^x)^2\,dx = \int \left(1+2e^x+e^{2x}\right)\,dx = x+2e^x+\frac{1}{2}e^{2x}+C$$

Split Fractions

$$\begin{aligned}\int \frac{1+x}{x^2+1}\,dx &= \int \left(\frac{1}{x^2+1}+\frac{x}{x^2+1}\right)\,dx \\ &= \int \frac{1}{x^2+1}\,dx + \frac{1}{2}\int \frac{1}{u}\,du \quad \boxed{\text{substitute: } \begin{array}{l} u = x^2+1 \\ du = 2x\,dx \end{array}} \\ &= \tan^{-1}x + \frac{1}{2}\ln|u| + C = \tan^{-1}x + \frac{1}{2}\ln\left(x^2+1\right)+C\end{aligned}$$

Add and Subtract an Expression

$$\begin{aligned}\int \frac{2x}{x^2+2x+1}\,dx &= \int \frac{2x+2-2}{x^2+2x+1} \quad \boxed{\text{add and subtract 2}} \\ &= \int \frac{2x+2}{x^2+2x+1}\,dx - 2\int \frac{1}{(x+1)^2}\,dx \quad \boxed{\text{split the fraction}} \\ &= \int \frac{1}{u}\,du - 2\int \frac{1}{v^2}\,dv \quad \boxed{\text{substitute: } \begin{array}{l} u = x^2+2x+1 \\ du = (2x+2)dx \\ \text{and} \\ v = x+1 \\ dv = dx \end{array}} \\ &= \ln|u| + \frac{2}{v} + C = \ln\left(x^2+2x+1\right) + \frac{2}{x+1} + C\end{aligned}$$

Complete the Square

Example 1:
$$\begin{aligned}\int \frac{1}{\sqrt{2x-x^2}}\,dx &= \int \frac{1}{\sqrt{1-(x^2-2x+1)}}\,dx \quad \boxed{\begin{array}{l}\text{add and subtract 1 to} \\ \text{complete the square}\end{array}} \\ &= \int \frac{1}{\sqrt{1-(x-1)^2}}\,dx \\ &= \int \frac{1}{\sqrt{1-u^2}}\,du \quad \boxed{\text{substitute: } \begin{array}{l} u = x-1 \\ du = dx \end{array}} \\ &= \sin^{-1}u + C = \sin^{-1}(x-1)+C\end{aligned}$$

Example 2:
$$\begin{aligned}\int \frac{1}{4x^2-2x+9}\,dx &= \int \frac{1}{\left(4x^2-2x+\frac{1}{4}\right)-\frac{1}{4}+9}\,dx \quad \boxed{\begin{array}{l}\text{add and subtract } \frac{1}{4} \text{ to} \\ \text{complete the square}\end{array}} \\ &= \int \frac{1}{\left(2x-\frac{1}{2}\right)^2+\frac{35}{4}}\,dx \\ &= \frac{4}{35}\int \frac{1}{\left(\frac{4}{\sqrt{35}}x-\frac{1}{\sqrt{35}}\right)^2+1}\,dx \quad \boxed{\text{substitute: } \begin{array}{l} u = \frac{4}{\sqrt{35}}x-\frac{1}{\sqrt{35}} \\ du = \frac{4}{\sqrt{35}}dx \end{array}} \\ &= \frac{1}{\sqrt{35}}\int \frac{1}{u^2+1}\,du = \frac{1}{\sqrt{35}}\tan^{-1}u + C \\ &= \frac{1}{\sqrt{35}}\tan^{-1}\left(\frac{4}{\sqrt{35}}x-\frac{1}{\sqrt{35}}\right)+C\end{aligned}$$

Trick	**Examples**
Divide Improper Rational Fractions	$\int \frac{x^2}{x^2+1}\,dx = \int \left(1 - \frac{1}{x^2+1}\right)\,dx$ long division or synthetic division $= x - \tan^{-1} x + C$
Trig Identities	$\int \cos^2 x\,dx = \int \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)\,dx$ half-angle formula: $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$ $= \frac{1}{2}x + \frac{1}{4}\sin 2x + C$
Multiply and Divide	$\int \frac{1}{1+\sin x}\,dx = \int \frac{1}{1+\sin x}\cdot\frac{1-\sin x}{1-\sin x}\,dx$ multiply and divide by $1-\sin x$ $= \int \frac{1-\sin x}{1-\sin^2 x}\,dx$ expand denominator $= \int \frac{1-\sin x}{\cos^2 x}\,dx$ Pythagorean's Theorem: $\cos^2\theta + \sin^2\theta = 1$ $= \int \frac{1}{\cos^2 x}\,dx - \int \frac{\sin x}{\cos^2 x}\,dx$ split the fraction $= \int \sec^2 x\,dx - \int \tan x \sec x\,dx$ $= \tan x - \sec x + C$