Improper Integrals

1. An integral is improper if the integrand, the domain of integration, or both, are unbounded.

2. If \( f(x) \) is continuous on \([a, b)\) and \( f(x) \to \pm \infty \) as \( x \to b^- \), then by definition,
\[
\int_a^b f(x) \, dx = \lim_{r \to b^-} \int_a^r f(x) \, dx. \tag{1}
\]

3. If \( f(x) \) is continuous on \([a, b)\) and \( f(x) \to \pm \infty \) as \( x \to b^- \), then by definition,
\[
\int_a^b f(x) \, dx = \lim_{r \to b^-} \int_a^r f(x) \, dx. \tag{2}
\]

4. If \( f(x) \) is continuous on \([a, c) \cup (c, b]\), and unbounded at \( x = c \), then
\[
\int_a^b f(x) \, dx = \lim_{r \to c^-} \int_a^r f(x) \, dx + \lim_{s \to c^+} \int_s^b f(x) \, dx. \tag{3}
\]

The integral on the left-hand side of (3) is convergent if and only if the two on the right-hand side are.

5. If \( f(x) \) is continuous on \([a, \infty)\), then by definition,
\[
\int_a^\infty f(x) \, dx = \lim_{r \to \infty} \int_a^r f(x) \, dx. \tag{4}
\]

6. If \( f(x) \) is continuous on \((-\infty, b]\), then by definition,
\[
\int_{-\infty}^b f(x) \, dx = \lim_{r \to -\infty} \int_r^b f(x) \, dx. \tag{5}
\]

7. If \( f(x) \) is continuous on \((-\infty, \infty)\), then by definition,
\[
\int_{-\infty}^\infty f(x) \, dx = \lim_{r \to -\infty} \int_r^c f(x) \, dx + \lim_{s \to \infty} \int_s^c f(x) \, dx, \tag{6}
\]
where \( c \) is any fixed point. The integral on the left-hand side of (6) is convergent if and only if the two on the right-hand side are.

8. For example, if \( f(x) \) is continuous on \((a, b)\) and \( f(x) \to \pm \infty \) as \( x \to a^+ \) and \( x \to b^- \), then we define
\[
\int_a^b f(x) \, dx = \lim_{r \to a^+} \int_r^c f(x) \, dx \lim_{s \to b^-} \int_c^s f(x) \, dx, \tag{7}
\]
where \( c \) is any point in \((a, b)\). The integral on the left-hand side of (7) is convergent if and only if the two on the right-hand side are.