The Gradient

1. The gradient of $f : \mathbf{R}^3 \mapsto \mathbf{R}$ is the vector

$$\nabla f = \langle f_x, f_y, f_z \rangle\,, \tag{1}$$

with the obvious modification for functions of two variables. If, for example,

$$f(x,y,z) = x^2y + y^2z,$$

then

$$\nabla f(x,y,z) = \left\langle 2xy, x^2 + 2yz, y^2 \right\rangle, \tag{2}$$

and

$$\nabla f(1,1,2) = \langle 2,4,1 \rangle\,. \tag{3}$$

2. The derivative of f at the point P in the unit direction $\vec{u}$ is

$$D_{\vec{u}} f(P) = \nabla f(P) \cdot \vec{u}. \tag{4}$$

If $\vec{v}$ is not a unit vector, then *by definition*, the derivative of f at P in the direction $\vec{v}$ is $D_{\vec{u}} f(P)$, where $\vec{u} = \vec{v}/|\vec{v}\,|$. Note that the directional derivative is a scalar. You can think of it as the rate of change of f at P in the direction $\vec{v}$. If f is a function of two variables, the directional derivative is the slope of the surface $z = f(x,y)$ at the point P, in the direction $\vec{v}$.

3. We used formula (4) to show that $\nabla f(P)$ is the direction of steepest ascent, or most rapid increase, of the function f at the point P. In other words, $D_{\vec{u}} f(P)$ is maximized when

$$\vec{u} = \frac{\nabla f(P)}{|\nabla f(P)|}.$$

That steepest slope, or maximum rate of increase is $|\nabla f(P)|$. By the same token, the direction and rate of most rapid descent are

$$\vec{u} = -\frac{\nabla f(P)}{|\nabla f(P)|},$$

$-\|\nabla f(P)\|$ respectively.

4. Suppose a surface $\mathcal{S}$ is given implicitly by the equation

$$F(x,y,z) = K, \tag{5}$$

for some C^1 function F and a constant K. You can think of $\mathcal{S}$ as

a. a level surface of F, i.e. the surface on which $F(x,y,z)$ has the constant value K, or

b. the graph of a function $z = z(x,y)$, defined implicitly by equation (5).

As we saw in class, if (a,b,c) lies on $\mathcal{S}$, then the vector $\nabla F(a,b,c)$ is normal to $\mathcal{S}$ at that point. So, for example, the sphere $\mathcal{S}$ of radius $\sqrt{6}$ centered at the origin is given by

$$F(x,y,z) = x^2 + y^2 + z^2 = 6. \tag{6}$$

The vector

$$\nabla F(1,2,-1) = \langle 2,4,-2 \rangle\,,$$

is normal to the sphere at $(1,2,-1)$.

5. If $\mathcal{S}$ is the graph of a function given explicitly by

$$z = f(x, y), \tag{7}$$

we set

$$F(x, y, z) = f(x, y) - z,$$

and obtain the implicit representation

$$F(x, y, z) = 0. \tag{8}$$

According to the last paragraph, the vector

$$\nabla F(a, b, c) = \langle f_x(a, b), f_y(a, b), -1 \rangle\,,$$

is normal to the graph at the point (a, b, c), where $c = f(a, b)$. For example, the surface $\mathcal{S}$ given explicitly by $z = f(x, y) = x^2 - 2y^2$, has implicit form

$$F(x, y, z) = x^2 - 2y^2 - z = 0.$$

Hence the vector

$$\nabla F(3, 1, 7) = \langle 6, -4, -1 \rangle\,,$$

is normal to the graph at $(3, 1, 7)$.

6. The analogous statement holds for curves. Let $\mathcal{C}$ be a curve given implicitly by the equation

$$\nabla G(x, y) = K.$$

If (a, b) is a point on $\mathcal{C}$, then $\nabla G(a, b)$ is normal to $\mathcal{C}$ at that point.